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On the Negative Pell equation $y^2 = 21x^2 - 5$

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Abstract
The binary quadratic equation represented by the negative Pellian $y^2 = 21x^2 - 5$ is analyzed for its distinct integer solutions. A few interesting relations among the solutions are given. Further, employing the solutions of the above hyperbola, we have obtained solutions of other choices of hyperbola, parabola.

Keywords: binary quadratic, hyperbola, parabola, pell equation, integral solutions

Introduction

A binary quadratic equation of the form $y^2 = Dx^2 + 1$, where D is non-square positive integer has been studied by various mathematicians for its non-trivial integral solutions when D takes different integral values [1-2]. For an extensive review of various problems, one may refer [3-15]. In this communication, yet another interesting hyperbola given by $y^2 = 21x^2 - 5$ considered and infinitely many integer solutions are obtained. A few interesting properties among the solutions are obtained. Further, employing the solutions of the above hyperbola, we have obtained solutions of other choices of hyperbola, parabola.

Method of analysis

The Negative Pell equation representing hyperbola under consideration is

$$y^2 = 21x^2 - 5 \tag{1}$$

whose smallest positive integer solution is

$$x_0 = 1,\ y_0 = 4$$

To obtain the other solutions of (1), consider the Pell equation

$$y^2 = 21x^2 + 1$$

whose general solution is given by

$$\tilde{x}_n = \frac{1}{2\sqrt{21}} g_n,\ \tilde{y}_n = \frac{1}{2} f_n$$

where

$$f_n = \left(55 + 12\sqrt{21}\right)^{n+1} + \left(55 - 12\sqrt{21}\right)^{n+1}$$

$$g_n = \left(55 + 12\sqrt{21}\right)^{n+1} - \left(55 - 12\sqrt{21}\right)^{n+1},\ n = -1, 0, 1, 2......$$

Applying Brahmagupta lemma between (x_0, y_0) and $(\tilde{x}_n, \tilde{y}_n)$, the other integer solutions of (1) are given by

$$x_{n+1} = \frac{1}{2} f_n + \frac{2}{\sqrt{21}} g_n$$

$$y_{n+1} = 2 f_n + \frac{\sqrt{21}}{2} g_n$$

The recurrence relations satisfied by x and y are given by

$$x_{n+3} - 110x_{n+2} + x_{n+1} = 0$$

$$y_{n+3} - 110y_{n+2} + y_{n+1} = 0$$

Some numerical examples of x and y satisfying (1) are given in the Table: 1 below:

Table 1: Numerical examples

n	x_{n+1}	y_{n+1}
-1	1	4
0	103	472
1	11329	51916
2	1246087	5710288
3	137058241	628079764
4	15075160423	69083063752

From the above table, we observe some interesting relations among the solutions which are presented below:

1. x_{n+1} is always odd and y_{n+1} is always even
2. Relations among the solutions

- $55x_{n+1} - x_{n+2} + 12y_{n+1} = 0$
- $x_{n+1} - 55x_{n+2} + 12y_{n+2} = 0$
- $55x_{n+1} - 6049x_{n+2} + 12y_{n+3} = 0$
- $6049x_{n+1} - x_{n+3} + 1320y_{n+1} = 0$
- $6049x_{n+2} - 55x_{n+3} + 12y_{n+1} = 0$
- $55x_{n+2} - x_{n+3} + 12y_{n+2} = 0$
- $x_{n+1} - x_{n+3} + 24y_{n+2} = 0$
- $x_{n+1} - 6049x_{n+3} + 1320y_{n+3} = 0$
- $x_{n+2} - 55x_{n+3} + 12y_{n+3} = 0$
- $55y_{n+1} - y_{n+2} + 252x_{n+1} = 0$
- $y_{n+1} - 55y_{n+2} + 252x_{n+2} = 0$
- $55y_{n+1} - 6049y_{n+2} + 252x_{n+3} = 0$
- $6049y_{n+1} - y_{n+3} + 27720x_{n+1} = 0$
- $y_{n+1} - y_{n+3} + 504x_{n+2} = 0$
- $y_{n+1} - 6049y_{n+3} + 27720x_{n+3} = 0$
- $6049y_{n+2} - 55y_{n+3} + 252x_{n+1} = 0$
- $55y_{n+2} - y_{n+3} + 252x_{n+2} = 0$
- $y_{n+2} - 55y_{n+3} + 252x_{n+3} = 0$

3. Each of the following expressions represents a nasty number

- $\frac{1}{5}\left(472x_{2n+2} - 4x_{2n+3} + 60\right)$
- $\frac{2}{275}\left(12979x_{2n+2} - x_{2n+4} + 1650\right)$
- $\frac{12}{5}\left(21x_{2n+2} - 4y_{2n+2} + 5\right)$

- $\frac{12}{275}\left(2163x_{2n+2} - 4y_{2n+3} + 275\right)$
- $\frac{12}{30245}\left(237909x_{2n+2} + 4y_{2n+4} + 30245\right)$
- $\frac{4}{5}\left(12979x_{2n+3} - 118x_{2n+4} + 15\right)$
- $\frac{12}{275}\left(21x_{2n+3} - 472y_{2n+2} + 275\right)$
- $\frac{12}{5}\left(2163x_{2n+3} - 472y_{2n+3} + 5\right)$
- $\frac{12}{275}\left(237909x_{2n+3} - 472y_{2n+4} + 275\right)$
- $\frac{12}{30245}\left(21x_{2n+4} - 51916y_{2n+2} + 30245\right)$
- $\frac{12}{275}\left(2163x_{2n+4} - 51916y_{2n+3} + 275\right)$
- $\frac{12}{5}\left(237909x_{2n+4} - 51916y_{2n+4} + 5\right)$
- $\frac{1}{5}\left(y_{2n+3} - 103y_{2n+2} + 60\right)$
- $\frac{1}{550}\left(y_{2n+4} - 11329y_{2n+2} + 6600\right)$
- $\frac{1}{5}\left(103y_{2n+4} - 11329y_{2n+3} + 60\right)$

4. Each of the following expressions represents a cubical integer

- $\frac{1}{30}\left[472x_{3n+3} - 4x_{3n+4} + 1416x_{n+1} - 12x_{n+2}\right]$
- $\frac{1}{3300}\left[51916x_{3n+3} - 4x_{3n+5} + 155748x_{n+1} - 12x_{n+3}\right]$
- $\frac{1}{5}\left[42x_{3n+3} - 8y_{3n+3} + 126x_{n+1} - 24y_{n+1}\right]$
- $\frac{1}{275}\left[4326x_{3n+3} - 8y_{3n+4} + 12978x_{n+1} - 24y_{n+2}\right]$
- $\frac{1}{30245}\left[475818x_{3n+3} - 8y_{3n+5} + 1427454x_{n+1} - 24y_{n+3}\right]$
- $\frac{1}{30}\left[51916x_{3n+4} - 472x_{3n+5} + 155748x_{n+2} - 1416x_{n+3}\right]$
- $\frac{1}{275}\left[42x_{3n+4} - 944y_{3n+3} + 126x_{n+2} - 2832y_{n+1}\right]$
- $\frac{1}{5}\left[4326x_{3n+4} - 944y_{3n+4} + 12978x_{n+2} - 2832y_{n+2}\right]$
- $\frac{1}{275}\left[475818x_{3n+4} - 944y_{3n+5} + 1427454x_{n+2} - 2832y_{n+3}\right]$
- $\frac{1}{30245}\left[42x_{3n+5} - 103832y_{3n+3} + 126x_{n+3} - 311496y_{n+1}\right]$

- $\frac{1}{275}\left[4326x_{3n+5}-103832y_{3n+4}+12978x_{n+3}-311496y_{n+2}\right]$
- $\frac{1}{5}\left[475818x_{3n+5}-103832y_{3n+5}+1427454x_{n+3}-311496y_{n+3}\right]$
- $\frac{1}{30}\left[y_{3n+4}-103y_{3n+3}+3y_{n+2}-309y_{n+1}\right]$
- $\frac{1}{3300}\left[y_{3n+5}-11329y_{3n+3}+3y_{n+3}-33987y_{n+1}\right]$
- $\frac{1}{30}\left[103y_{3n+5}-11329y_{3n+4}+309y_{n+3}-33987y_{n+2}\right]$

5. Each of the following expressions represents a bi-quadratic integer

- $\frac{1}{30}\left[472x_{4n+4}-4x_{4n+5}+1888x_{2n+2}-16x_{2n+3}+180\right]$
- $\frac{1}{3300}\left[51916x_{4n+4}-4x_{4n+6}+207664x_{2n+2}-16x_{2n+4}+19800\right]$
- $\frac{1}{5}\left[42x_{4n+4}-8y_{4n+4}+168x_{2n+2}-32y_{2n+2}+30\right]$
- $\frac{1}{275}\left[4326x_{4n+4}-8y_{4n+5}+17304x_{2n+2}-32y_{2n+3}+1650\right]$
- $\frac{1}{30245}\left[475818x_{4n+4}-8y_{4n+6}+1903272x_{2n+2}-32y_{2n+4}+181470\right]$
- $\frac{1}{30}\left[51916x_{4n+5}-472x_{4n+6}+207664x_{2n+3}-1888x_{2n+4}+180\right]$
- $\frac{1}{275}\left[42x_{4n+5}-944y_{4n+4}+168x_{2n+3}-3776y_{2n+2}+1650\right]$
- $\frac{1}{5}\left[4326x_{4n+5}-944y_{4n+5}+17304x_{2n+3}-3776y_{2n+3}+30\right]$
- $\frac{1}{275}\left[475818x_{4n+5}-944y_{4n+6}+1903272x_{2n+3}-3776y_{2n+4}+1650\right]$
- $\frac{1}{30245}\left[42x_{4n+6}-103832y_{4n+4}+168x_{2n+4}-415328y_{2n+2}+181470\right]$
- $\frac{1}{275}\left[4326x_{4n+6}-103832y_{4n+5}+17304x_{2n+4}-415328y_{2n+3}+1650\right]$
- $\frac{1}{5}\left[475818x_{4n+6}-103832y_{4n+6}+1903272x_{2n+4}-415328y_{2n+4}+30\right]$
- $\frac{1}{30}\left[y_{4n+5}-103y_{4n+4}+4y_{2n+3}-412y_{2n+2}+180\right]$
- $\frac{1}{3300}\left[y_{4n+6}-11329y_{4n+4}+4y_{2n+4}-45316y_{2n+2}+19800\right]$
- $\frac{1}{30}\left[103y_{4n+6}-11329y_{4n+5}+412y_{2n+4}-45316y_{2n+3}+180\right]$

6. Each of the following expressions represents a quintic integer

- $\frac{1}{30}\left[472x_{5n+5}-4x_{5n+6}+2360x_{3n+3}-20x_{3n+4}+4720x_{n+1}-40x_{n+2}\right]$

- $\frac{1}{3300}\begin{bmatrix} 51916x_{5n+5} - 4x_{5n+7} + 259580x_{3n+3} - 20x_{3n+5} \\ + 519160x_{n+1} - 40x_{n+3} \end{bmatrix}$
- $\frac{1}{5}[42x_{5n+5} - 8x_{5n+5} + 210x_{3n+3} - 40y_{3n+3} + 420x_{n+1} - 80y_{n+1}]$
- $\frac{1}{275}\begin{bmatrix} 4326x_{5n+5} - 8y_{5n+6} + 21630x_{3n+3} - 40y_{3n+4} + 43260x_{n+1} \\ - 80y_{n+2} \end{bmatrix}$
- $\frac{1}{30245}\begin{bmatrix} 475818x_{5n+5} - 8y_{5n+7} + 2379090x_{3n+3} - 40y_{3n+5} \\ + 4758180x_{n+1} - 80y_{n+3} \end{bmatrix}$
- $\frac{1}{30}\begin{bmatrix} 51916x_{5n+6} - 472x_{5n+7} + 259580x_{3n+4} - 2360x_{3n+5} \\ + 519160x_{n+2} - 4720x_{n+3} \end{bmatrix}$
- $\frac{1}{275}\begin{bmatrix} 42x_{5n+6} - 944y_{5n+5} + 210x_{3n+4} - 4720y_{3n+3} + 420x_{n+2} \\ - 9440y_{n+1} \end{bmatrix}$
- $\frac{1}{5}\begin{bmatrix} 4326x_{5n+6} - 944y_{5n+6} + 21630x_{3n+4} - 4720y_{3n+4} \\ + 43260x_{n+2} - 8440y_{n+2} \end{bmatrix}$
- $\frac{1}{275}\begin{bmatrix} 475818x_{5n+6} - 944y_{5n+7} + 2379090x_{3n+4} - 4720y_{3n+5} \\ + 4758180x_{n+2} - 9440y_{n+3} \end{bmatrix}$
- $\frac{1}{275}\begin{bmatrix} 4326x_{5n+7} - 103832y_{5n+6} + 21630x_{3n+5} - 519160y_{3n+4} \\ + 43260x_{n+3} - 1038320y_{n+2} \end{bmatrix}$
- $\frac{1}{5}\begin{bmatrix} 475818x_{5n+7} - 103832y_{5n+7} + 2379090x_{3n+5} - 519160y_{3n+5} \\ + 4758180x_{n+3} - 1038320y_{n+3} \end{bmatrix}$
- $\frac{1}{30}[y_{5n+6} - 103y_{5n+5} + 5y_{3n+4} - 515y_{3n+3} + 10y_{n+2} - 1030y_{n+1}]$
- $\frac{1}{3300}\begin{bmatrix} y_{5n+7} - 11329y_{5n+5} + 5y_{3n+5} - 56645y_{3n+3} + 10y_{n+3} \\ - 113290y_{n+1} \end{bmatrix}$
- $\frac{1}{30}\begin{bmatrix} 103y_{5n+7} - 11329y_{5n+6} + 515y_{3n+5} - 56645y_{3n+4} + 1030y_{n+3} \\ - 113290y_{n+2} \end{bmatrix}$
- $\frac{1}{30245}\begin{bmatrix} 42x_{5n+7} - 103832y_{5n+5} + 210x_{3n+5} - 519160y_{3n+3} \\ + 420x_{n+3} - 1038320y_{n+1} \end{bmatrix}$

Remarkable observations

1. Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbola which are presented in the Table: 2 below:

Table 2: Hyperbolas

S. No	Hyperbolas	(X, Y)
1	$X^2 - 21Y^2 = 3600$	$(472x_{n+1} - 4x_{n+2}, x_{n+2} - 103x_{n+1})$
2	$X^2 - 21Y^2 = 43560000$	$(51916x_{n+1} - 4x_{n+1}, x_{n+3} - 11329x_{n+1})$
3	$X^2 - 21Y^2 = 100$	$(42x_{n+1} - 8y_{n+1}, 2y_{n+1} - 8x_{n+1})$
4	$X^2 - 21Y^2 = 302500$	$(4326x_{n+1} - 8y_{n+2}, 2y_{n+2} - 944x_{n+1})$
5	$X^2 - 21Y^2 = 3659040100$	$(475818x_{n+1} - 8y_{n+3}, 2y_{n+3} - 103832x_{n+1})$
6	$X^2 - 21Y^2 = 3600$	$(51916x_{n+2} - 472x_{n+3}, 103x_{n+3} - 11329x_{n+2})$

7	$X^2 - 21Y^2 = 302500$	$(42x_{n+2} - 944y_{n+1}, 206y_{n+1} - 8x_{n+2})$
8	$X^2 - 21Y^2 = 100$	$(4326x_{n+2} - 944y_{n+2}, 206y_{n+2} - 944x_{n+2})$
9	$X^2 - 21Y^2 = 302500$	$(475818x_{n+2} - 944y_{n+3}, 206y_{n+3} - 103832x_{n+2})$
10	$X^2 - 21Y^2 = 3659040100$	$(42x_{n+3} - 103832y_{n+1}, 22658y_{n+1} - 8x_{n+3})$
11	$X^2 - 21Y^2 = 302500$	$(4326x_{n+3} - 103832y_{n+2}, 22658y_{n+2} - 944x_{n+3})$
12	$X^2 - 21Y^2 = 100$	$(475818x_{n+3} - 103832y_{n+3}, 22658y_{n+3} - 103832x_{n+3})$
13	$18900X^2 - 900Y^2 = 68040000$	$(y_{n+2} - 103y_{n+1}, 472y_{n+1} - 4y_{n+2})$
14	$21X^2 - Y^2 = 914760000$	$(y_{n+3} - 11329y_{n+1}, 51916y_{n+1} - 4y_{n+3})$
15	$21X^2 - Y^2 = 75600$	$(103y_{n+3} - 11329y_{n+2}, 51916y_{n+2} - 472y_{n+3})$

2. Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of parabola which are presented in the Table: 3 below:

Table 3: Parabolas

S. No	Parabolas	(X, Y)
1	$30X - 21Y^2 = 1800$	$(472x_{2n+2} - 4x_{2n+3}, x_{n+2} - 103x_{n+1})$
2	$3300X - 21Y^2 = 21780000$	$(5196x_{2n+2} - 4x_{2n+4}, x_{n+3} - 11329x_{n+1})$
3	$5X - 21Y^2 = 50$	$(42x_{2n+2} - 8y_{2n+2}, 2y_{n+1} - 8x_{n+1})$
4	$275X - 21Y^2 = 151250$	$(4326x_{2n+2} - 8y_{2n+3}, 2y_{n+2} - 944x_{n+1})$
5	$30245X - 21Y^2 = 1829520050$	$(475818x_{2n+2} - 8y_{2n+4}, 2y_{n+3} - 103832x_{n+1})$
6	$30X - 21Y^2 = 1800$	$(51916x_{2n+3} - 472x_{2n+4}, 103x_{n+3} - 11329x_{n+2})$
7	$275X - 21Y^2 = 151250$	$(42x_{2n+3} - 944y_{2n+2}, 206y_{n+1} - 8x_{n+2})$
8	$5X - 21Y^2 = 50$	$(4326x_{2n+3} - 944y_{2n+3}, 206y_{n+2} - 944x_{n+2})$
9	$275X - 21Y^2 = 151250$	$(475818x_{2n+3} - 944y_{2n+4}, 206y_{n+3} - 103832x_{n+2})$
10	$30245X - 21Y^2 = 1829520050$	$(42x_{2n+4} - 103832y_{2n+2}, 22658y_{n+1} - 8x_{n+3})$
11	$275X - 21Y^2 = 151250$	$(4326x_{2n+4} - 103832y_{2n+3}, 22658y_{n+2} - 944x_{n+3})$
12	$5X - 21Y^2 = 50$	$(475818x_{2n+4} - 103832y_{2n+4}, 22658y_{n+3} - 103832x_{n+3})$
13	$18900X - 30Y^2 = 1134000$	$(y_{2n+3} - 103y_{2n+2}, 472y_{n+1} - 4y_{n+2})$
14	$69300X - Y^2 = 457380000$	$(y_{2n+4} - 11329y_{2n+2}, 51916y_{n+1} - 4y_{n+3})$
15	$630X - Y^2 = 37800$	$(103y_{2n+4} - 11329y_{2n+3}, 51916y_{n+2} - 472y_{n+3})$

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