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Coincidence and common fixed points of two self-mappings in fuzzy metric spaces

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Abstract

In this work, the coincidence and common fixed points of two self-maps under weakly compatible condition on fuzzy metric space are proved.

Keywords: φ −contractions, coincidence point, common fixed point

1. Introduction

The concept of fuzzy set was introduced in the pioneering paper of Zadah [8]. Thereafter, Grabiec [1] defined the G-complete metric space and proved some fixed point results on the fuzzy metric space. Mishra *et al.* [4] also gave several fixed point theorems for asymptotically commutating maps in the same space. George and Veeramani [5] modified the definition of the Cauchy sequence given by Grabiec [1]. The definition of fuzzy metric space introduced by Kramosil and Michalek [2] also modified. Since then, this modified definition given by George and Veeramani is today known as M-Complete fuzzy metric space. Every G-complete metric space is M-complete fuzzy metric space.

Many results of common fixed points have been given in metric spaces, fuzzy metric spaces, partial metric spaces etc. As Banach contraction principle is indeed a classical result of modern analysis which is of great importance. This result is extended in many spaces i.e. fuzzy metric space, partial metric space, Menger metric space, cone metric space etc. Likewise, in [9] a common fixed point of two mappings such that one mapping is ψ – weak contraction with respect to another mapping has been given.

First, some definitions which are used to prove the main results.

Definition1.1 [6]

A binary operation $*:[0, 1]\times [0,1] \to [0,1]$is called a continuous t-norm if it satisfies the following conditions:

(TN-1) $*$ is commutative and associative;
(TN-2) $*$ is continuous;
(TN-3) $a * 1 = a$ for every $a \in [0, 1]$;
(TN-4) $a * b \le c * d$ whenever $a \le c, b \le d$ and $a, b, c, d \in [0, 1]$.

Definition 1.2 [5]

A fuzzy metric space is an ordered triple (X, M, $*$) such that X is a (nonempty) set, $*$ is a continuous t-norm and M is a fuzzy set on $X\times X \times (0,1)$ satisfying the following conditions, for all $x, y, z \in X$, $t > 0$:

(FM -1) $M (x, y, t) > 0$;
(FM -2) $M(x, y, t) = 1$ if and only if $x = y$;
(FM -3) $M(x, y, t) = M (y, x, t)$;
(FM -4) $M(x, y, t) * M (y, z, s)$ £ $M(x, z, t + s)$;
(FM -5) $M(x, y, .): (0,\infty) \to (0,1)$ is continuous.

Definition 1.3 [5, 3]

Let (X, M, $*$) be a fuzzy metric space. Then:

1) A sequence $\{x_n\}$ is said to converge to x in X, denoted by $x_n \to x$, if and only if $\lim_{n\to\infty} M(x_n, x, t)$ for all $t > 0$, i.e. for each $r \in (0,1)$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x, t) > 1 - r$ for all $n \ge n_0$.
2) A sequence $\{x_n\}$ in X is an M-Cauchy sequence if and only if for each
a. $0 <\in< 1$, $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $M(x_m, x_n, t) > 1- \in$ for any m, $n \ge n_0$.
3) The fuzzy metric space (X, M, $*$) is called M-complete if every M-Cauchy sequence is convergent in X.

Definition 1.4 [1]

A sequence $\{x_n\}$ in a fuzzy metric space (X, M, $*$) is G-Cauchy if $\lim_{n\to+\infty} (M_{n+p} , x_n, t)$ for every $t > 0$ and for every $p > 0$.

Definition 1.5 [9]

Let (X, M, $*$) be a fuzzy metric space and f, T be two self mappings on X. A point x in X is called a coincidence point (common fixed point) of f and T if fx = Tx (fx = Tx = x). Also the pair of mappings f,T:X→X are said to be weakly compatible if they commute on the set of coincidence points.

Definition 1.6 [7]
The function φ: [0, 1] → $[0, 1]$ which is used by altering the distance between two points satisfies the following properties:
(P1) φ is strictly decreasing and left continuous;
(P2) φ (λ) = 0 if and only if $\lambda = 1$.
Obviously, we obtain that $\lim_{\lambda \to 1^-} \varphi(\lambda) = \varphi(1) = 0$.

Theorem 1.7 [7]
Let (X, M, ∗) be an M-complete fuzzy metric space and T a self-map of X and suppose that φ:[0, 1] → $[0, 1]$ satisfies the foregoing properties (P1) and (P2). Furthermore, let k be a function from $(0, \infty)$ into (0, 1). If for any t > 0, T satisfies the following condition:

$$\varphi\big(M(Tx, Ty, t)\big) \le \text{k(t)}\, \varphi(M(x, y, t))$$

Where $x, y \in X$ and $x \neq y$, then T has a unique fixed point.

2. Coincidence and common fixed points of two self-mappings
In this section, two theorems are proved. The first result establishes coincidence point of two self-mappings on fuzzy metric spaces and second result establishes a unique common fixed point of two self-mappings on fuzzy metric spaces.

Theorem 2.1
Let (X, M, ∗) be a fuzzy metric space and suppose φ:[0, 1] → $[0, 1]$ satisfies the foregoing properties (P1) and (P2). Furthermore, let k be a function from (0, ∞) to (0, 1). If T and f are two self maps of X satisfying

$$\varphi\big(M(Tx, Ty, t)\big) \le \text{k(t)}\, \varphi(M(\text{f}x, \text{f}y, t)) \qquad (2.1)$$

Where $x, y \in X$ and $x \neq y$ and f(X) is a G-complete subspace of X, then f and T have coincidence point in X.

Proof
Let x_0 be an arbitrary point in X. Let $x_1 \in X$ such that $Tx_0 = fx_1$. This is possible since the range of f contains the range of T. Continuing, this process indefinitely, for every $x_n \in X$, one can find a x_{n+1} such that $y_n = Tx_n = fx_{n+1}$. Without loss of generality, one may assume that $y_{n+1} \neq y_n$ for all $n \in N$, otherwise f and T have a coincidence point and there is nothing to prove. In case, $y_{n+1} \neq y_n$ from (2.1), we have

$$\begin{aligned}\varphi\big(M(y_n, y_{n+1}, t)\big) &= \text{k(t)}\, \varphi(M(Tx_n, Ty_n, t)) \\ &\le \text{k(t)}\, \varphi(M(fx_n, fx_{n+1}, t)) \\ &= \text{k(t)}\, \varphi(M(y_{n-1}, y_n, t)) \\ &< \varphi(M(y_{n-1}, y_n, t)).\end{aligned}$$

This implies

$$\varphi\big(M(y_n, y_{n+1}, t)\big) < \varphi(M(y_{n-1}, y_n, t)). \qquad (2.2)$$

But, φ is strictly decreasing function. This implies that

$$\varphi\big(M(y_n, y_{n+1}, t)\big) > \varphi(M(y_{n-1}, y_n, t)) \text{ for all n} \in N.$$

This implies that $\{M(y_n, y_{n+1}, t)\}$ is increasing sequence of positive real numbers in [0, 1]. Let $S(t) = \lim_{n\to\infty} M(y_n, y_{n+1}, t)$. Now, we show that S(t) = 1 for all t > 0. Otherwise, there must exist some t > 0 such that S(t) < 1. Taking n → ∞ in (2.2), we obtain φ(S(t)) < φ(S(t)), a contradiction.

Therefore
$$M(y_n, y_{n+1}, t) = 1 \text{ as n} \to \infty.$$

Note that, for each positive integer p,

$$M\big(y_n, y_{n+p}, t\big) \ge M(y_n, y_{n+1}, t/p) * M(y_{n+1}, y_{n+2}, t/p) \ldots M\big(y_{n+p-1}, y_{n+p}, t/p\big).$$

This implies that
$$\lim_{n\to\infty} M\big(y_n, y_{n+p}, t\big) \ge 1 * 1 * \ldots * 1 = 1.$$

Therefore $\{y_n\}$ is a G-Cauchy sequence. Since f(X) is G-complete, therefore there exists $q \to f(X)$ such that $y_n \to q$ as n → ∞. consequently, we obtain p in X such that fp = q. Next we show that p is coincidence point of f and T. Now using (2.1), we have

$$\varphi\big(M(Tp,\ fx_{n+1},t)\big) = k(t)\ \varphi(M(Tp,Tx_n,t))$$
$$\leq k(t)\ \varphi(M(fp,fx_n,t)).$$

This implies,

$$\lim_{n\to\infty} \varphi\big(M(Tp,\ fx_{n+1},t)\big) \leq \lim_{n\to\infty} k(t)\ \varphi(M(fp,fx_n,t)).$$

This implies,

$$0 \leq \varphi\big(M(Tp,fp,t)\big) \leq k(t)\ \varphi(M(fp,fp,t)) = k(t)\ \varphi(1) = 0.$$

So $\varphi\big(M(Tp,fp,t)\big) = 0.$

So by property (P2) of φ, we have

M(Tp, fp, t) = 1

This implies Tp = fp. This completes the proof the theorem.

Theorem 2.2

Let (X, M, ∗) be a fuzzy metric space and suppose φ:[0, 1] → $[0,1]$ satisfies foregoing properties (P1) and (P2). Furthermore, let k be a function from (0, ∞) into (0, 1). If T and f are two self maps of X satisfying:

$$\varphi\big(M(Tx,Ty,t)\big) \leq k(t)\ \varphi(M(fx,fy,t)) \qquad (2.3)$$

where $x, y \in X$ and $x \neq y$ and f(X) is a G-complete subspace of X, then f and T have a unique common fixed point provided that the pair of mappings {f, T} is weakly compatible.

Proof

By theorem 2.1, we obtain a point p in X such that Tp = fp = q (say) which further implies fTp = Tfp, since f and T are weakly compatible. Obviously Tq = fq. Now we show that fq = q, If not, then

$$\varphi\big(M(fq,q,t)\big) = k(t)\ \varphi(M(Tq,Tp,t))$$
$$\leq k(t)\ \varphi(M(fq,fp,t))$$
$$= k(t)\ \varphi\big(M(fq,q,t)\big). \qquad (2.4)$$

This implies that

$$\varphi(M(fq,q,t) = 0.$$

So by properties (P1) of φ , we obtain

M (fq, q, t) = 1.

This implies fq = q.
Now we prove uniqueness of the theorem. Let y be another common fixed point of f and T i.e. fy = Ty = y.
Now

$$\varphi\big(M(y,q,t)\big) = k(t)\ \varphi(M(Ty,Tp,t))$$
$$\leq k(t)\ \varphi(M(fy,fp,t))$$
$$< \varphi\big(M(y,q,t)\big)$$

a contradiction which proves y = q. This completes the proof of the theorem.

Example 2.3

Let X = [0, 1] and a ∗ b = min{a, b}. Let M be the standard fuzzy metric space induced by d where d(x, y) = |x –y| for x, y ∈ X. Then (X, M, ∗) is the complete fuzzy metric space. Let f (t) = $1-\sqrt{t}$ for all t ∈ [0, 1] and define k:(0, ∞)→ (0, 1) as

$$k(t) = \begin{cases} 1 - e^{-4/t} & 0 < t \leq 2 \\ \frac{t}{t+1} & t > 2 \end{cases}$$

Also, let

$$fx = \frac{1}{2}(1 - x),\ x \in [0, 1]$$

and

$$Tx = \frac{1}{3}, \text{ for all } x \in [0, 1]$$

Then f and T satisfy all the conditions of the theorem 2.2. Note that 1/3 is the coincidence point which also turns out to be common fixed point also.

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