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# Coincidence and common fixed point theorems for (φ, ψ): Weak contractions of two self-maps in fuzzy metric spaces

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**Abstract**

By using (φ, ψ)-weak contractions, the existence of coincidence point and common fixed point of two self-maps in the fuzzy metric space is established.

**Keywords:** (φ, ψ)-weak contractions, coincidence point, common fixed point

## 1. Introduction

The introduction to fuzzy sets was given by Zadeh [14] in 1965. Its applications can be seen in many branches of applied science namely neutral networking theory, image processing control theory etc. The notion of fuzzy metric was introduced in [3]. George and Veeramani [7, 8] modified the definition of a fuzzy metric space which was firstly introduced by Kramosil and Michalek [2]. In this space, Banach contraction principle was extended and generalized in different ways. Grabiec [5] gave many fixed points in fuzzy metric space (see also [6], [4], [10]) thereafter, many fixed point results were established by using contractive mappings in modified fuzzy metric spaces. Rhoades [9] proved a fixed point theorem for ψ–weak contraction in fuzzy metric space and M. Abbas, M. Imdad, and D. Gopal [15] also established a common fixed point theorem for ψ–weak contraction in fuzzy metric space.

The following definitions and results are used to prove the main results.

**Definition 1.1** [14]

A fuzzy set A in a nonempty set X is a function with domain X and values in [0, 1].

**Definition 1.2** [12]

A binary operation $*: [0, 1] \times [0, 1] \rightarrow [0, 1]$ is a continuous $t -$ norm if it satisfies the following conditions:

(1) $*$ is associative and commutative;
(2) $*$ is continuous;
(3) $a * 1 = a$ for every $a \in [0, 1]$;
(4) $a * b \leq c * d$ if $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0, 1]$.

**Definition 1.3** [7]

The triplet $(X, M, *)$ is a fuzzy metric space (in the sense of George and Veeramani) if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set in $X \times X \times [0, 1)$satisfying the following conditions:

(i) $M(x, y, t) > 0$,
(ii) $M(x, y, t) = 1$ iff $x = y$,
(iii) $M(x, y, t) = M(y, x, t)$,
(iv) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,
(v) $M(x, y, .) : (0, \infty) \rightarrow [0, 1]$ is continuous for each $x, y, z \in X$ and $s, t > 0$.

Note that, M (x, y, t) can be realized as the measure of nearness between x and y with respect to t. It is known that M (x, y, .) is nondecreasing for all $x, y \in X$. George and Veeramani [7, 8] proved that every fuzzy metric M on X induces a Hausdorff first countable topology $t_M$ whose base is the family of open balls, $\{B(x, r, t), x \in X, 0 < r < 1, t > 0\}$, where

$$B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}$$

**Definition 1.4** [15]

A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ is said to be a Cauchy sequence if for every $0 < \epsilon < 1$ and for every $t > 0$, there is $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \epsilon$ for every $n, m \geq n_0$.

**Definition 1.5** [15]

A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ is said to be a G−Cauchy sequence (i.e. Cauchy sequence in the sense of Grabiec [5]) if $M(x_n, x_{n+p}, t) \rightarrow 1$ as $n \rightarrow \infty$, for every $p \in \mathbb{N}$ and for every $t > 0$. Hence, a fuzzy metric space $(X, M, *)$ is called complete (respectively G-complete) if every Cauchy sequence (respectively G−Cauchy sequence) is convergent. Vasuki and Veeramani suggested that the definition of a G−Cauchy sequence is weaker than the one contained in [13].

**Definition 1.6** [15]
Let $(X, M, *)$ be a fuzzy metric space and f, T be two self mappings on X. A point x in X is called a coincidence point (common fixed point) of f and T if $fx = Tx$ ($fx = Tx = x$). Also the pair of mappings $f, T:X \to X$ are said to be weakly compatible if they commute on the set of coincidence points.

**Example 1.7** [8]
If d is a metric on X, $a*b = ab$ for all $a, b \in [0, 1]$ and $M(x,y,t) = \frac{t}{t+d(x,y)}$ for every $(x, y, t) \in X{\times}X{\times}[0, 1)$ then $(X, M, *)$ is a fuzzy metric space. We call this M as the standard fuzzy metric induced by d. Even if we define $a*b = \min\{a, b\}$, $(X, M, *)$ will be a fuzzy metric space.

**2. Coincidence and common fixed point results of two self-maps in fuzzy metric space**
Here, the main results of coincidence and common fixed point results of two self-maps in fuzzy metric spaces are presented.

**Theorem 2.1**
Let $(X, M, *)$ be a fuzzy metric space and T and f are the self-maps on X such that

$$\varphi(M(Tx, Ty, t)) \leq \varphi(M(fx, fy, t)) - \psi(M(fx, fy, t)) \qquad (2.1)$$

Where $\psi:[0, \infty) \to [0, \infty)$ is continuous with $\psi(r) > r$ if $r > 0$ and $\psi(0) = 0$ and $\varphi:[0, 1] \to [0, 1]$ is continuous, strictly decreasing and left continuous with $\varphi(\lambda) = 0$ if and only if $\lambda = 1$ . If the range of f contains the range of T and f(X) is a G-complete subspace of X, then f and T have coincidence point in X.

**Proof**
Let $x_0$ be an arbitrary point in X. Choose a point $x_1$ in X such that $Tx_0 = fx_1$. This is so possible because the range of f contains the range of T. Continuing this process indefinitely, for every $x_n$ in X one can find a $x_{n+1}$ such that $y_n = Tx_n = fx_{n+1}$. If $y_n = y_{n+1}$ for some n Î N , then f and T have a coincidence point and there is nothing to prove. So let $y_n \neq y_{n+1}$ for all $n \in N$.
Now using (2.1), we have

$$\begin{aligned}\varphi(M(y_n, y_{n+1}, t)) &= \varphi(M(Tx_n, Tx_{n+1}, t))\\ &\leq \varphi(M(fx_n, fx_{n+1}, t)) - \psi(fx_n, fx_{n+1}, t))\\ &< \varphi(M(fx_n, fx_{n+1}, t))\\ &= \varphi(M(y_{n-1}, y_n, t)) \qquad (2.2)\end{aligned}$$

Since φ is strictly decreasing, which implies

$$M(y_n, y_{n+1}, t) > M(y_{n-1}, y_n, t)$$

For all n and hence $M(y_{n-1}, y_n, t)$ is an increasing sequence of positive real numbers in (0, 1]. Let $S(t) = \lim_{n\to\infty} M(y_{n-1}, y_n, t)$. Now we show $S(t) = 1$ for all $t > 0$. Otherwise there must exist some $t > 0$ such that $S(t) < 1$. Taking $n \to \infty$, in (2.2), we obtain

$$\varphi(S(t)) \leq \varphi(S(t)) - \psi(S(t))$$

Which is a contradiction. Therefore $M(y_n, y_{n+1}, t) \to 1$ as $n \to \infty$. Note that, for each positive integer p,

$$M(y_n, y_{n+p}, t) \geq M(y_n, y_{n+1}, t/p) * M(y_{n+1}, y_{n+2}, t/p) * \ldots * M(y_{n+p-1}, y_{n+p}, t/p).$$

This implies that

$$\lim_{n\to\infty} M(y_n, y_{n+p}, t) \geq 1 * 1 * 1 \cdots * 1 = 1$$

Therefore $\{y_n\}$ is a G-Cauchy sequence. Since f(X) is G-Complete, there exists $q \in f(X)$ such that $y_n \to q$ *as* $n \to \infty$. consequently, we obtain a point p in X such that $fp = q$. Next we show that p is a coincidence point of f and T. using (2.1), we obtain

$$\begin{aligned}\varphi(M(Tp, fx_{n+1}, t)) &= \varphi(M(Tp, Tx_n, t))\\ &\leq \varphi(M(fp, fx_n, t)) - \psi(fp, fx_n, t)) \text{ for every } t > 0.\\ &< \varphi(M(fp, fx_n, t))\end{aligned}$$

Taking $n \to \infty$, we obtain

$$0 \leq \varphi(M(Tp, fp, t)) \leq \varphi(1) = 0.$$

This implies $M(Tp, fp, t) = 1$ for all $t > 0$. So we obtain $Tp = fp$. This completes the proof the theorem.

**Theorem 2.2**

Let $(X, M, *)$ be a fuzzy metric space and T and f are the self-maps on X satisfying

$$\varphi(M(Tx, Ty, t)) \leq \varphi(M(fx, fy, t)) - \psi(M(fx, fy, t))$$

Where $\psi:[0, \infty) \to [0, \infty)$ is continuous with $\psi(r) > r$ if $r > 0$ and $\psi(0) = 0$ and $\varphi:[0, 1] \to [0, 1]$ is continuous , strictly decreasing and left continuous with $\varphi(\lambda) = 0$ if and only if $\lambda = 1$. If the range of f contains the range of T and f(X) is a G − Complete subspace of X, Then f and T have a unique common fixed point in X provided that f and T are compatible mappings.

**Proof**

By Theorem (2.1), we obtain a point p in X such that $Tp = fp = q$ (say). Which further implies $fTp = Tfp$ since f and T are weakly compatible. Obviously, $Tq = fq$. Now we show that $fq = q$. If not, then

$$\begin{aligned} \varphi(M(fq, q, t)) &= \varphi(M(Tq, Tp, t)) \\ &\leq \varphi(M(fq, fp, t)) - \psi(fq, fp, t)) \\ &= \varphi(M(fq, q, t)) - \psi(fq, q, t)) \end{aligned}$$

This implies

$$\varphi(M(fq, q, t)) < \varphi(M(fq, q, t))$$

Which is a contradiction. Hence $fq = q$. Therefore f and T have a common fixed point in X. Now we prove the uniqueness of the theorem. Let $y$ be another common fixed point of f and T. i.e. $Ty = fy = y$

Now

$$\begin{aligned} \varphi(M(q, y, t)) = \varphi(M(fq, fy, t)) &= \varphi(M(Tq, Ty, t)) \\ &\leq \varphi(M(fq, fy, t)) - \varphi(M(fq, fy, t)) \\ &= \varphi(M(q, y, t)) = \varphi(M(q, y, t)) \\ &< \varphi(M(q, y, t)) \end{aligned}$$

a contradiction and this proves the uniqueness of the theorem. i. e. $q = y$.

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