International Journal of Advanced Science and Research
ISSN: 2455-4227, Impact Factor: RJIF 5.12
www.allsciencejournal.com
Volume 2; Issue 1; January 2017; Page No. 18-22

On THEs binary quadratic equation $y^2=7x^2+32$

[1] K Meena, [2] S Vidhyalakshmi, [3] R Sobana Devi

[1] Former VC, Bharathidasan University, Trichy, Tamil Nadu, India.
[2] Professor, Department of Mathematics, SIGC, Trichy, Tamil Nadu, India.
[3] M.Phil Scholar, Department of Mathematics, SIGC, Trichy, Tamil Nadu, India.

Abstract
The Binary Quadratic Equation represented by the positive Pellian $y^2=7x^2+32$ is analyzed for its distinct integer solutions. A few interesting among the solutions are given Further, employing the solutions of the above hyperbola, we have obtained solutions of other choices of hyperbolas, parabolas and special pythagorian triangle.

Keywords: binary quadratic, parabola, integral solution, pell equation

Introduction
The binary quadratic equation of the form $y^2 = Dx^2 + 32$ where, D is non-square positive integer has been studied by various mathematicians for its non-trivial integral solutions when D takes different integral values [1-4]. For an extensive review of various problems, one may refer [5-20]. In this communication, yet another interesting hyperbola given by $y^2 = 7x^2 + 32$ is considered and infinitely many integer solutions are obtained. A few interesting properties among the solutions are presented.

Method of analysis
The positive pell equation representing hyperbola under consideration is,

$$y^2 = 7x^2 + 32 \qquad (1)$$

The smallest positive integer solution is,

$$x_0 = 4, \; y_0 = 12$$

The general solution (x_n, y_n) of (1) is given by,

$$y_n = \frac{1}{2} f_n, \quad x_n = \frac{1}{2\sqrt{7}} g_n \qquad (2)$$

Where,

$$f_n = (8+3\sqrt{7})^{n+1} + (8-3\sqrt{7})^{n+1}$$

$$g_n = (8+3\sqrt{7})^{n+1} - (8-3\sqrt{7})^{n+1}$$

The recurrence relations satisfied by the solution (2) are given by,

$$x_{n+3} - 16x_{n+2} + x_{n+1} = 0$$

$$y_{n+3} - 16y_{n+2} + y_{n+1} = 0, \quad n = 0,1,2,.....$$

Some numerical examples of x and y satisfying (1) are given in the Table 1 below.

Table 1: Examples

n	x_n	y_n
0	4	12
1	68	180
2	1084	2868
3	17276	45708
4	275332	728460

Form the above table, we observe some interesting relations among the solutions which are presented below.

1. The x_n values are even.

2. The y_n values are always even.

3. Each of the following expressions is a nasty numbers:

- $\frac{1}{2}(17y_{2n-2} - y_{2n+3} + 24)$
- $\frac{1}{16}(y_{2n+1} - 357x_{2n+3} + 192)$
- $\frac{1}{256}(9y_{2n+4} - 5673y_{2n+2} + 3048)$
- $\frac{1}{2}(9x_{2n+3} - 45x_{2n+2} + 24)$
- $\frac{1}{32}(3x_{2n+4} - 717x_{2n+2} + 384)$
- $\frac{1}{16}(135y_{2n+2} - 21x_{2n+3} + 192)$
- $\frac{1}{2}(135y_{2n+3} - 357x_{2n+3} + 24)$
- $\frac{1}{16}(135y_{2n+4} - 5691x_{2n+3} + 192)$
- $\frac{1}{2}(45x_{2n+4} - 717x_{2n+3} + 24)$
- $\frac{1}{254}(2151y_{2n+2} - 21x_{2n+4} + 3048)$
- $\frac{1}{16}(2151y_{2n+3} - 357x_{2n+4} + 192)$
- $\frac{1}{2}(2151y_{2n+4} - 5691x_{2n+4} - 24)$
- $\frac{1}{32}(y_{2n+3} - 271y_{2n+2} - 384)$
- $\frac{1}{2}(12y_{2n+3} - 271y_{2n+3} + 24)$

4. Each of the following expressions is a cubical integer:

- $144(17y_{3n+3} - y_{3n+4}) + 432(17y_{n+1} - y_{n+2})$
- $1024(3y_{3n+4} - 119x_{3n+3}) + 3072(3y_{n+2} - 119x_{n+1})$
- $2(x_{3n+4} - 15x_{3n+3}) + 6(x_{n+2} - 15x_{n+1})$
- $8(x_{3n+5} - 239x_{3n+3}) + 24(x_{n+3} - 239x_{n+1})$
- $1024(45y_{3n+3} - 7x_{3n+4}) + 3072(45y_{n+1} - 7x_{n+2})$
- $2(45y_{3n+4} - 119x_{3n+4}) + 6(45y_{n+2} - 119x_{n+2})$
- $1024(45y_{3n+5} - 1897x_{3n+4}) + 3072(45y_{n+3} - 1897x_{n+2})$
- $144(45x_{3n+5} - 717x_{3n+4}) + 432(45x_{n+3} - 717x_{n+2})$
- $258064(717y_{3n+3} - 7x_{3n+5}) + 774192(717y_{n+1} - 7x_{n+3})$
- $1024(717y_{3n+4} - 119x_{3n+5}) + 3072(717y_{n+2} - 119x_{n+3})$
- $2(717y_{3n+5} - 1897x_{3n+5}) + 6(717y_{n+3} - 1897x_{n+3})$
- $36864(271y_{3n+3} - y_{3n+5}) + 110592(271y_{n+1} - y_{n+3})$
- $144(271y_{3n+4} - 17y_{3n+5}) + 432(271y_{n+3} - 17y_{n+3})$

5. Relation among the solutions

- $21x_{n+1} = y_{n+2} - 8y_{n+1}$
- $8x_{n+1} = x_{n+2} - 3y_{n+1}$
- $x_{n+1} = 8x_{n+2} - 3y_{n+2}$

- $x_{n+1} = 16x_{n+3} - 255x_{n+2}$
- $127x_{n+1} = x_{n+3} - 48y_{n+1}$
- $x_{n+1} = x_{n+3} - 6y_{n+1}$
- $x_{n+1} = 127x_{n+3} - 48y_{n+3}$
- $336x_{n+1} = y_{n+3} - 127y_{n+1}$
- $21x_{n+1} = 8y_{n+3} - 127y_{n+2}$
- $8x_{n+1} = 127x_{n+2} - 3y_{n+3}$
- $21x_{n+2} = 8y_{n+2} - y_{n+1}$
- $x_{n+2} = x_{n+2} - 254x_{n+1}$
- $127x_{n+2} = 8x_{n+3} - 3y_{n+1}$
- $8x_{n+2} = x_{n+3} - 3y_{n+2}$
- $x_{n+2} = 8x_{n+3} - 3y_{n+3}$
- $42x_{n+2} = y_{n+3} - y_{n+1}$
- $21x_{n+2} = y_{n+3} - 8y_{n+2}$
- $21x_{n+3} = 127y_{n+2} - 8y_{n+1}$
- $x_{n+3} = 16x_{n+2} - x_{n+1}$
- $336x_{n+3} = 127y_{n+3} - y_{n+1}$
- $21x_{n+3} = 8y_{n+3} - y_{n+2}$
- $6y_{n+1} = x_{n+3} - 253x_{n+1}$
- $3y_{n+1} = 127x_{n+3} - 2024x_{n+2}$
- $y_{n+1} = 16y_{n+2} - y_{n+3}$
- $254y_{n+2} = 85281x_{n+1} - 103y_{n+3}$
- $48y_{n+2} = 127x_{n+3} - 32257x_{n+1}$
- $3y_{n+2} = 2024x_{n+3} - 32257x_{n+2}$
- $6y_{n+3} = 253x_{n+3} - 64261x_{n+1}$
- $3y_{n+3} = 32257x_{n+3} - 514088x_{n+2}$

Remarkable observations

i) Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbolas which are presented in the Table 2 below.

Table 2: Hyperbolas

S. No.	(X,Y)	HYPERBOLA
1	$(17y_{n+1} - y_{n+2}, y_{n+2} - 15y_{n+1})$	$7X^2 - 9Y^2 = 4032$
2	$(3y_{n+2} - 119x_{n+1}, y_{n+2} - 45x_{n+1})$	$X^2 - Y^2 = 4096$
3	$(3y_{n+3} - 1897x_{n+1}, y_{n+3} - 717x_{n+1})$	$X^2 - Y^2 = 1032256$
4	$(x_{n+2} - 15x_{n+1}, x_{n+2} - 17x_{n+1})$	$9X^2 - Y^2 = 576$
5	$(x_{n+3} - 239x_{n+1}, x_{n+3} - 271x_{n+1})$	$9X^2 - 7Y^2 = 1032192$
6	$(45y_{n+1} - 7x_{n+2}, 17y_{n+1} + 2x_{n+2})$	$X^2 - Y^2 = 4096$
7	$(45y_{n+2} - 119x_{n+2}, 17y_{n+2} - 45x_{n+2})$	$X^2 - Y^2 = 64$
8	$(y_{n+3} - 1897x_{n+2}, 17y_{n+3} - 717x_{n+2})$	$X^2 - Y^2 = 4624$

9	$(45x_{n+3}-717x_{n+2},17x_{n+3}-271x_{n+2})$	$X^2-Y^2=576$
10	$(717y_{n+1}-7x_{n+3},271y_{n+1}-3x_{n+3})$	$X^2-Y^2=1032256$
11	$(717y_{n+2}-119x_{n+3},271y_{n+2}-45x_{n+2})$	$X^2-Y^2=4096$
12	$(717y_{n+3}-1897x_{n+3},271y_{n+3}-717x_{n+3})$	$X^2-Y^2=64$
13	$(y_{n+3}-271y_{n+1},y_{n+3}-239y_{n+1})$	$7X^2-9Y^2=1032192$
14	$(17y_{n+3}-271y_{n+2},45y_{n+3}-717y_{n+2})$	$7X^2-Y^2=4032$

ii) Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of parabolas which are presented in the Table 2 below.

Table 3: Parabolas

S. No.	(X,Y)	PARABOLA
1	$(y_{2n+3}-17y_{2n+2}-24,y_{n+2}-15y_{n+1})$	$3Y^2=-28X+1344$
2	$(3y_{2n+1}-119x_{2n+2}+64,y_{n+2}+45x_{n+1})$	$Y^2=X-4096$
3	$(3y_{2n+4}-717x_{2n+2}+1016,y_{n+3}-717x_{n+1})$	$Y^2=508X-1032256$
4	$(x_{2n+3}-15x_{2n+2}+8,17x_{n+1}-x_{n+2})$	$Y^2=36X-576$
5	$(x_{2n+4}-239x_{2n+2}+128,x_{n+3}-271x_{n+1})$	$7Y^2=576X-147456$
6	$(45y_{2n+2}-7x_{2n+3}+64,2x_{n+2}-17y_{n+1})$	$Y^2=32X-4096$
7	$(45y_{2n+3}-119x_{2n+3}+8,17y_{n+2}-45x_{n+2})$	$Y^2=4X-64$
8	$(45y_{2n+4}-1897x_{2n+3}+64,17y_{n+3}-717x_{n+2})$	$Y^2=32X-4096$
9	$(45x_{2n+4}-717x_{2n+3}+24,17x_{n+3}-271x_{n+2})$	$Y^2=12X-576$
10	$(717y_{2n+2}-7x_{2n+4}+1016,271y_{n+1}-3x_{n+3})$	$Y^2=508X-1032256$
11	$(717y_{2n+3}-119x_{2n+4}+64,271y_{n+2}-45x_{n+3})$	$Y^2=32X-4096$
12	$(717y_{2n+4}-1897x_{2n+4}-8,271y_{n+3}-714x_{n+3})$	$Y^2=4X-64$
13	$(y_{2n+3}-271y_{2n+2}-384,y_{n+3}-239y_{n+1})$	$3Y^2=-448X-344064$
14	$(12y_{2n+3}-271y_{2n+3}+24,45y_{n+3}-717y_{n+2})$	$Y^2=84X-4032$

iii) Consider $m=x_{n+1}+y_{n+1},n=x_{n+1}$ observe that $m>n>0$. Treat m,n as the generators of the Pythagorean triangle $T(\alpha,\beta,\gamma)$ where,

$\alpha=2pq$, $\beta=p^2-q^2$, $\gamma=p^2+q^2$

Then the following interesting relations are observed.

A. $2\alpha-7\beta+5\gamma=-64$

B. $9\alpha-2\gamma=\frac{28A}{P}-64$

C. $\frac{2A}{P}=x_{n+1}y_{n+1}$

Conclusion

In this paper, we have presented infinitely many integer solutions for the hyperbola represented by the positive pell equation $y^2=7x^2+32$. As the binary quadratic Diophantine equations are rich is variety, one many search for the other choices of positive pell equations and determine their integer solutions along with suitable properties.

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