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Sum of finite terms of Pell sequence in terms of NSW sequence and interrelationship of some sequences

[1] Mihir B Trivedi, *[2] Dr. Pradeep J Jha

[1] Department of Mathematics, Research scholar, Rai University, Ahmedabad, Gujarat, India
[2] Department of Mathematics, Rai University, Ahmedabad, Gujarat, India

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Abstract

Varied Contents of this note are the projections on introducing different sequences and their characteristics. It is also interesting to note the salient features of right triangles of Fermat family and inter connection of its sides to that of the corresponding terms of the NSW* sequence. A run on using proper algebra helps derive a new sequence also. The elegant feature is to organise the set of positive integers in three mutually disjoint infinite classes and finally working out with sum of Pell sequence. Generating functions of the sequences are derived in annexure.

Keywords: fermat family, jha sequence, pell sequence, hypotenuse sequence, sum of pell sequence

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1. Introduction to Right triangles of Fermat family

A right triangle F_i $where\ i = 1,2,3, \ldots$ is said to be the i^{th} member of Fermat family if for some $a_i \in N \cup \{0\}$ $and\ b_i, and\ h_i \in N$ with the following conditions are satisfied:

1. $a_i < b_i < h_i$
2. $(a_i, b_i) = 1$
3. $b_i - a_i = 1$
4. $a_i^{\ 2} + b_i^{\ 2} = h_i^{\ 2}$ ∀ i ∈ N ….. (1)

We shall call a_i – shorter leg of Right triangle F_i

We denote the infinite set of all such right triangles of Fermat family by F.

$$F = \{F_i | F_i = F_i(a_i, b_i, h_i)\ \forall i \in N\} \qquad \text{….. (2)}$$

Table 1: Each F_i satisfies all four above mentioned properties. We identify some members of the set F

$\boldsymbol{F_i}$ **Triangle Number**	$\boldsymbol{a_i}$ **Shorter Leg**	$\boldsymbol{b_i}$ **Next Leg**	$\boldsymbol{h_i}$ **Hypotenuse**
F_1	$a_1 = 0$ *	$b_1 = 1$	$h_1 = 1$
F_2	$a_2 = 3$	$b_2 = 4$	$h_2 = 5$
F_3	$a_3 = 20$	$b_3 = 21$	$h_3 = 29$
F_4	$a_4 = 119$	$b_4 = 120$	$h_4 = 169$
F_5	$a_5 = 696$	$b_5 = 697$	$h_5 = 985$

[*We have considered $a_i = 0\ for\ i = 1$ and justify the same with given above mentioned conditions. In addition to that it satisfies all the properties of triangle.]

2. Introduction to some Known sequences and its generalization

2.1 Jha Sequence

Table 2: In continuation with the above Table1, we construct the following table.

$\boldsymbol{F_i}$ **Triangle Number**	$\boldsymbol{S_i = (a_i + b_i + h_i)/2}$	$\Delta_i = \boldsymbol{A_i = \frac{1}{2} a_i b_i}$	$\boldsymbol{r_i = \frac{\Delta_i}{S_i}}$
F_1	1	0	0
F_2	6	6	1
F_3	35	210	6
F_4	204	7140	35
F_5	1189	242556	204

It is clearly observed that

a) Semi perimeter (S_i) of i^{th} triangle is the in-radius (r_i) of $(i+1)^{th}$ triangle.

b) The terms of successive $F_i's$ observes recurrence relation

$r_{i+2} = 6r_{i+1} - r_i$, $i \in\ N\ with\ r_1 = 0\ and\ r_2 = 1$. Each term is called a Jha number and sequence is called a Jha sequence.

Some terms $(J_n, n = 1,2,3 \ldots)$ of Jha Sequence are 0, 1, 6, 35, 204, 1189… ….. (3)
This permits us to write the its recurrence relation as follows

$$J_{n+2} = 6J_{n+1} - J_n \; Where \; n \in N \; with \; J_1 = 0 \; and \; J_2 = 1 \quad \ldots..(4)$$

Using the recurrence relation we derive its general term.

$$J_n = \frac{\sqrt{2}}{8}\left[(3+2\sqrt{2})^{n-1} - (3-2\sqrt{2})^{n-1}\right] \; where \; n \in N \quad \ldots..(5)$$

The generating function is given by,

$$J(x) = \frac{x}{1-6x+x^2} \quad \ldots..(6)$$

2.2 Pell Sequence

Pell numbers most notably appear in the infinite sequence that converge to the better approximation to the irrational number $\sqrt{2}$. We write the infinite set of pairs of positive integers $x \; and \; y \; i.e. (x, y)$ which contains the solution of the equation

$$x^2 - 2y^2 = \pm 1$$

The trivial solution of the above equation is $(1,0)$ and the remaining solutions are

$$\{(1,1), (3,2), (7,5), (17,12), (41,29), (99,70), \ldots\}$$

We denote each member of the above infinite set as pair$(x_i, y_i), i \in N$.

i.e. $(x_1, y_1) = (1,0), (x_2, y_2) = (1,1), (x_3, y_3) = (3,2), (x_4, y_4) = (7,5), \ldots$

We observe that second coordinate of each pair is a Pell number(P_n). In other words the coordinate y_i of the pair (x_i, y_i) represents the Pell number for all values of $i \in N$ and if we take ratio
$\frac{x_i}{y_i}, i \in N - \{1\}$, then its value provides better approximation of$\sqrt{2}$ as $i \to \infty$
It gives relatively a better mathematical felling that,

$$\lim_{i\to\infty}\left(\frac{x_i}{y_i}\right) = \sqrt{2}$$

Some terms$(P_n, n = 1,2,3 \ldots)$ of the Pell sequence are 0, 1, 2, 5, 12, 29, 70 … ….. (7)
From this, we write the recurrence relation as follows

$$P_{n+2} = 2P_{n+1} + P_n \; Where \; n \in N \; with \; , P_1 = 0, P_2 = 1 \quad \ldots..(8)$$

Using the recurrence relation we derive its general term as,

$$P_n = P(n) = \frac{\sqrt{2}}{4}\left[(1+\sqrt{2})^{n-1} - (1-\sqrt{2})^{n-1}\right] \; where \; n \in N \quad \ldots..(9)$$

The generating function is given by,

$$P(x) = \frac{x}{1-2x+x^2} \quad \ldots..(10)$$

2.3 Sequence of Hypotenuse

While studying the Pell sequence, we observe that every even ordered Pell number serves as a hypotenuse of the triangle of the Fermat family.
In Section 2, Table 1 it is clearly seen that the terms of the last column are Pell numbers and they satisfies the recurrence relation

$$h_{n+2} = 6h_{n+1} - h_n \quad \ldots..(11)$$

$Where \; n \in N \; and \; h_n \; is \; the \; n^{th}$ term of sequence of hypotenuse $with \; h_1 = 1, h_2 = 5$
Some terms of the sequence of hypotenuse are 1, 5, 29, 169, 985… ….. (12)
Using the recurrence relation we derive its general term as,

$$h_n = \frac{1}{2\sqrt{2}}\left[\left(1+\sqrt{2}\right)(3+2\sqrt{2})^{n-1} + \left(-1+\sqrt{2}\right)(3-2\sqrt{2})^{n-1}\right] \text{ where } n \in N \qquad \text{….. (13)}$$

We can compress above general term using simple algebra, which can be stated as,

$$h_n = \frac{1}{2\sqrt{2}}\left[(1+\sqrt{2})^{2n-1} - (1-\sqrt{2})^{2n-1}\right] \text{ where } n \in N \qquad \text{….. (14)}$$

The generating function is given by,

$$h(x) = \frac{1-x}{1-6x+x^2} \qquad \text{….. (15)}$$

*Using the terms of Jha sequence the terms of sequence of hypotenuse can be obtained as follows.
Let J_n be the nth term of the Jha sequence then $J_1 = 0, J_2 = 1, J_3 = 6, J_4 = 35$ ...
By taking the difference of consecutive terms of Jha sequence we get,

$$J_2 - J_1 = 1 = h_1, J_3 - J_2 = 5 = h_2, J_4 - J_3 = 29 = h_3 \text{ ….}$$

In general it can be written as, $h_n = J_{n+1} - J_n$ $where\ n \in N\ and\ h_n\ is\ the\ n^{th}$ term of sequence of hypotenuse.
Also each term of this sequence is even ordered Pell number, so we can mention

$$h_n = P_{2n} \text{ where } n \in N$$

2.4 NSW Sequence

An NSW numbers named after Newman Shank and William is an integer that solves the Diophantine equation$2n^2 = m^2 + 1$. In other words NSW numbers (m) are indexing the diagonal of squares of side length n having the property that the square of the diagonal $d = \sqrt{2}\,n$ equals one plus a square number m^2.
The first few NSW numbers are therefore $m = 1, 7, 41, 239, 1393$ … , which correspond to square side length $n = 1, 5, 29, 169, 985, 5741$ …
NSW sequence can also be introduced as the sum of legs of the Fermat triangles.

Table 3: We construct the following table using the Table 1,

$\boldsymbol{n}$	$\boldsymbol{a_n}$	$\boldsymbol{b_n}$	$N_n = \boldsymbol{a_n} + \boldsymbol{b_n}$
1	0	1	1
2	3	4	7
3	20	21	41
4	119	120	239

Note that each element $(N_n, n = 1,2,3 \dots)$ of the last column of the above table satisfies the recurrence relation

$$N_{n+2} = 6N_{n+1} - N_n, n \in N \text{ with } N_1 = 1 \text{ and } N_2 = 7. \qquad \text{….. (16)}$$

Here each term is called NSW number and the sequence is known as NSW sequence.
Some terms of the sequence are 1, 7, 41, 239, 1393… ….. (17)
Using the recurrence relation we derive its general term as,

$$N_n = \frac{1}{2}\left[\left(1+\sqrt{2}\right)(3+2\sqrt{2})^{n-1} - \left(-1+\sqrt{2}\right)(3-2\sqrt{2})^{n-1}\right] \text{ where } n \in N \qquad \text{….. (18)}$$

The generating function is given by,

$$N(x) = \frac{1+x}{1-6x+x^2} \qquad \text{….. (19)}$$

*Using the terms of Jha sequence the terms of NSW sequence can be obtained as follows.
Let J_n be the nth term of the Jha sequence then $J_1 = 0, J_2 = 1, J_3 = 6, J_4 = 35$ ...
By adding the consecutive terms of Jha sequence we get,

$$J_2 + J_1 = 1 = N_1, J_3 + J_2 = 7 = N_2, J_4 + J_3 = 41 = N_3 \text{ ….}$$

In general it can be written as, $N_n = J_{n+1} + J_n$ $where\ n \in N\ and\ N_n\ is\ the\ n^{th}$ terms of NSW sequence.

3. Interrelated properties of these sequences

In above section we had a glance over the important sequences and their key features. Now we study the inter-connectivity of these sequences and important properties. In fact, we have identified and derived many mathematical properties. We enlist and prove them using their general terms.

1. $P_{2n} = J_{n+1} - J_n$
2. $h_n = J_{n+1} - J_n$
3. $h_{n+1} = h_n + 4\,J_{n+1}$
4. $(P_{n+1})^2 + (P_n)^2 = h_n$
5. $2P_{n+1} * P_n + (P_{n+1})^2 - (P_n)^2 = N_n$
6. Pythagorean Triplet$(a_n,\ b_n, h_n) = (2P_{n+1} * P_n\ ,\ (P_{n+1})^2 - (P_n)^2\ ,\ (P_{n+1})^2 +\ (P_n)^2)$
   $if\ n\ is\ an\ odd\ positive\ integer$

Pythagorean Triplet$(a_n\ , b_n\ , h_n) = ((P_{n+1})^2 - (P_n)^2\ , 2P_{n+1} * P_n\ , (P_{n+1})^2 + (P_n)^2)$
$if\ n\ is\ an\ even\ positive\ integer$

Proofs: We give proofs of the above mentioned results.

1 To prove $P_{2n} = J_{n+1} - J_n$

It is known that the general term of the Jha Sequence is

$J_n = \frac{\sqrt{2}}{8}\left[(3+2\sqrt{2})^{n-1} - (3-2\sqrt{2})^{n-1}\right]$; that help us derive

$$
\begin{aligned}
R.H.S = J_{n+1} - J_n &= \frac{\sqrt{2}}{8}\left[(3+2\sqrt{2})^{n} - \left(3-2\sqrt{2}\right)^{n} - \left(3+2\sqrt{2}\right)^{n-1} + (3-2\sqrt{2})^{n-1}\right] \\
&= \frac{\sqrt{2}}{8}\left[\left(3+2\sqrt{2}\right)^{n-1}\left(\left(3+2\sqrt{2}\right)-1\right) - \left(3-2\sqrt{2}\right)^{n-1}\left(\left(3-2\sqrt{2}\right)-1\right)\right] \\
&= \frac{\sqrt{2}}{4}\left[\left(3+2\sqrt{2}\right)^{n-1}\left(1+\sqrt{2}\right) - \left(3-2\sqrt{2}\right)^{n-1}\left(1-\sqrt{2}\right)\right] \\
&= \frac{\sqrt{2}}{4}\left[\left(3+2\sqrt{2}\right)^{n-1}\left(3+2\sqrt{2}\right)^{\frac{1}{2}} - \left(3-2\sqrt{2}\right)^{n-1}\left(3-2\sqrt{2}\right)^{\frac{1}{2}}\right] \\
&= \frac{\sqrt{2}}{4}\left[\left(3+2\sqrt{2}\right)^{n-\frac{1}{2}} - \left(3-2\sqrt{2}\right)^{n-\frac{1}{2}}\right] \\
&= \frac{\sqrt{2}}{4}\left[\left(1+\sqrt{2}\right)^{2n-1} - \left(1-\sqrt{2}\right)^{2n-1}\right] \\
&= P_{2n} \ = L.H.S
\end{aligned}
$$

This proves 1; $P_{2n} = J_{n+1} - J_n$

Next we prove the important result that connects the terms of hypotenuse sequence to that of Jha sequence.

2 To prove $h_n = J_{n+1} - J_n$ we use the general term of the Jha Sequence;

$$
\begin{aligned}
J_n &= \frac{\sqrt{2}}{8}\left[(3+2\sqrt{2})^{n-1} - (3-2\sqrt{2})^{n-1}\right] \\
R.H.S = J_{n+1} - J_n &= \frac{\sqrt{2}}{8}\left[(3+2\sqrt{2})^{n} - \left(3-2\sqrt{2}\right)^{n} - \left(3+2\sqrt{2}\right)^{n-1} + (3-2\sqrt{2})^{n-1}\right] \\
&= \frac{\sqrt{2}}{8}\left[\left(3+2\sqrt{2}\right)^{n-1}\left(\left(3+2\sqrt{2}\right)-1\right) - \left(3-2\sqrt{2}\right)^{n-1}\left(\left(3-2\sqrt{2}\right)-1\right)\right] \\
&= \frac{\sqrt{2}}{4}\left[\left(3+2\sqrt{2}\right)^{n-1}\left(1+\sqrt{2}\right) + \left(3-2\sqrt{2}\right)^{n-1}\left(\sqrt{2}-1\right)\right] \\
&= h_n \\
&= \mathrm{L.H.S}
\end{aligned}
$$

This proves 2; $h_n = J_{n+1} - J_n$

3 To prove $h_{n+1} = h_n + 4\,J_{n+1}$ we use the General terms of the Jha Sequence and sequence of Hypotenuse which read as follows.

$$
\begin{aligned}
J_n &= \frac{\sqrt{2}}{8}\left[(3+2\sqrt{2})^{n-1} - (3-2\sqrt{2})^{n-1}\right] \\
h_n &= \frac{1}{2\sqrt{2}}\left[\left(1+\sqrt{2}\right)(3+2\sqrt{2})^{n-1} + \left(-1+\sqrt{2}\right)(3-2\sqrt{2})^{n-1}\right] \\
R.H.S = h_n + 4\,J_{n+1} &= \frac{1}{2\sqrt{2}}\left[\left(1+\sqrt{2}\right)(3+2\sqrt{2})^{n-1} + \left(-1+\sqrt{2}\right)(3-2\sqrt{2})^{n-1}\right] + \frac{4\sqrt{2}}{8}\left[(3+2\sqrt{2})^{n} - (3-2\sqrt{2})^{n}\right] \\
&= \frac{1}{2\sqrt{2}}\left[\left(1+\sqrt{2}\right)(3+2\sqrt{2})^{n-1} + \left(-1+\sqrt{2}\right)(3-2\sqrt{2})^{n-1}\right] + \frac{1}{\sqrt{2}}\left[\left(3+2\sqrt{2}\right)(3+2\sqrt{2})^{n-1} + \left(3+2\sqrt{2}\right)(3-2\sqrt{2})^{n-1}\right]
\end{aligned}
$$

$$= \frac{1}{2\sqrt{2}}\begin{bmatrix}(1+\sqrt{2})(3+2\sqrt{2})^{n-1} + (-1+\sqrt{2})(3-2\sqrt{2})^{n-1} + \\ 2(1+\sqrt{2})^2(3+2\sqrt{2})^{n-1} - 2(1-\sqrt{2})^2(3-2\sqrt{2})^{n-1}\end{bmatrix}$$
$$= \frac{1}{2\sqrt{2}}\begin{bmatrix}(1+\sqrt{2})(3+2\sqrt{2})^{n-1}\left(1+2(1+\sqrt{2})\right) + \\ (\sqrt{2}-1)(3-2\sqrt{2})^{n-1}\left(1-2(\sqrt{2}-1)\right)\end{bmatrix}$$
$$= \frac{1}{2\sqrt{2}}\left[(1+\sqrt{2})(3+2\sqrt{2})^{n} + (-1+\sqrt{2})(3-2\sqrt{2})^{n}\right]$$
$$= h_{n+1}$$
$$= L.H.S$$

This proves 3; $h_{n+1} = h_n + 4\,J_{n+1}$

4 To prove $(P_{n+1})^2 + (P_n)^2 = h_n$ we use the General term of the Pell Sequence

$$P_n = P(n) = \frac{\sqrt{2}}{4}\left[(1+\sqrt{2})^{n-1} - (1-\sqrt{2})^{n-1}\right]$$
$$L.H.S = (P_{n+1})^2 + (P_n)^2$$
$$= \left(\frac{\sqrt{2}}{4}\left((1+\sqrt{2})^{n} - (1-\sqrt{2})^{n}\right)\right)^2 + \left(\frac{\sqrt{2}}{4}\left((1+\sqrt{2})^{n-1} - (1-\sqrt{2})^{n-1}\right)\right)^2$$
$$= \frac{1}{8}\left((1+\sqrt{2})^{2n} + (1-\sqrt{2})^{2n} - 2(1+\sqrt{2})^{n}(1-\sqrt{2})^{n}\right) + \frac{1}{8}\left((1+\sqrt{2})^{2n-2} + (1-\sqrt{2})^{2n-2} - 2(1+\sqrt{2})^{n-1}(1-\sqrt{2})^{n-1}\right)$$
$$= \frac{1}{8}\left((3+2\sqrt{2})^{n} + (3-2\sqrt{2})^{n} - 2(1+\sqrt{2})^{n}(1-\sqrt{2})^{n}\right) + \frac{1}{8}\left((3+2\sqrt{2})^{n-1} + (3-2\sqrt{2})^{n-1} - 2(1+\sqrt{2})^{n-1}(1-\sqrt{2})^{n-1}\right)$$
$$= \frac{1}{8}\left((3+2\sqrt{2})^{n-1}(4+2\sqrt{2}) + (3-2\sqrt{2})^{n-1}(4-2\sqrt{2}) - 2(1+\sqrt{2})^{n-1}(1-\sqrt{2})^{n-1}\left((1+\sqrt{2})(1-\sqrt{2})+1\right)\right)$$
$$= \frac{1}{4}\left((3+2\sqrt{2})^{n-1}(2+\sqrt{2}) + (3-2\sqrt{2})^{n-1}(2-\sqrt{2})\right)$$
$$= \frac{\sqrt{2}}{4}\left((3+2\sqrt{2})^{n-1}(\sqrt{2}+1) + (3-2\sqrt{2})^{n-1}(\sqrt{2}-1)\right)$$
$$= \frac{1}{2\sqrt{2}}\left((3+2\sqrt{2})^{n-1}(\sqrt{2}+1) + (3-2\sqrt{2})^{n-1}(\sqrt{2}-1)\right)$$
$$= h_n = R.H.S$$

This proves 4; $(P_{n+1})^2 + (P_n)^2 = h_n$

5 To prove $2P_{n+1} * P_n + (P_{n+1})^2 - (P_n)^2 = N_n$ we use the General Term of the Pell Sequence $P_n = P(n) = \frac{\sqrt{2}}{4}\left[(1+\sqrt{2})^{n-1} - (1-\sqrt{2})^{n-1}\right]$

First we show that,

$$2P_{n+1} * P_n = 2\left(\frac{\sqrt{2}}{4}\left((1+\sqrt{2})^{n} - (1-\sqrt{2})^{n}\right) * \frac{\sqrt{2}}{4}\left((1+\sqrt{2})^{n-1} - (1-\sqrt{2})^{n-1}\right)\right)$$
$$= \frac{1}{4}\left((1+\sqrt{2})^{2n-1} + (1-\sqrt{2})^{2n-1} - (1+\sqrt{2})^{n}(1-\sqrt{2})^{n-1} - (1+\sqrt{2})^{n-1}(1-\sqrt{2})^{n}\right)$$
$$= \frac{1}{4}\left((1+\sqrt{2})^{2n-1} + (1-\sqrt{2})^{2n-1} - (1+\sqrt{2})^{n-1}(1-\sqrt{2})^{n-1}(1+\sqrt{2}+1-\sqrt{2})\right)$$
$$= \frac{1}{4}\left((1+\sqrt{2})^{2n-1} + (1-\sqrt{2})^{2n-1} - 2(1+\sqrt{2})^{n-1}(1-\sqrt{2})^{n-1}\right) \qquad \text{…. (5.1)}$$

And then,

$$(P_{n+1})^2 - (P_n)^2$$
$$= \left(\frac{\sqrt{2}}{4}\left((1+\sqrt{2})^{n} - (1-\sqrt{2})^{n}\right)\right)^2 - \left(\frac{\sqrt{2}}{4}\left((1+\sqrt{2})^{n-1} - (1-\sqrt{2})^{n-1}\right)\right)^2$$
$$= \frac{1}{8}\left((1+\sqrt{2})^{2n} + (1-\sqrt{2})^{2n} - 2(1+\sqrt{2})^{n}(1-\sqrt{2})^{n}\right) - \frac{1}{8}\left((1+\sqrt{2})^{2n-2} + (1-\sqrt{2})^{2n-2} - 2(1+\sqrt{2})^{n-1}(1-\sqrt{2})^{n-1}\right)$$
$$= \frac{1}{8}\left((3+2\sqrt{2})^{n} + (3-2\sqrt{2})^{n} - 2(1+\sqrt{2})^{n}(1-\sqrt{2})^{n}\right) - \frac{1}{8}\left((3+2\sqrt{2})^{n-1} + (3-2\sqrt{2})^{n-1} - 2(1+\sqrt{2})^{n-1}(1-\sqrt{2})^{n-1}\right)$$
$$= \frac{1}{8}\left((3+2\sqrt{2})^{n-1}(2+2\sqrt{2}) + (3-2\sqrt{2})^{n-1}(2-2\sqrt{2}) - 2(1+\sqrt{2})^{n-1}(1-\sqrt{2})^{n-1}(-2)\right)$$
$$= \frac{1}{4}\left((3+2\sqrt{2})^{n-1}(1+\sqrt{2}) + (3-2\sqrt{2})^{n-1}(1-\sqrt{2}) + 2(1+\sqrt{2})^{n-1}(1-\sqrt{2})^{n-1}\right) \qquad \text{…. (5.2)}$$

And now Adding (5.1) and (5.2) we have,

$$\begin{aligned} L.H.S &= 2P_{n+1} * P_n + (P_{n+1})^2 - (P_n)^2 \\ &= \frac{1}{4}\left(\left(1+\sqrt{2}\right)^{2n-1} + \left(1-\sqrt{2}\right)^{2n-1} + \left(3+2\sqrt{2}\right)^{n-1}\left(1+\sqrt{2}\right) + \left(3-2\sqrt{2}\right)^{n-1}\left(1-\sqrt{2}\right)\right) \\ &= \frac{1}{2}\left(\left(1+\sqrt{2}\right)^{2n-1} + \left(1-\sqrt{2}\right)^{2n-1}\right) \\ &= \frac{1}{2}\left(\left(1+\sqrt{2}\right)\left(1+\sqrt{2}\right)^{2n-2} + \left(1-\sqrt{2}\right)\left(1-\sqrt{2}\right)^{2n-2}\right) \\ &= \frac{1}{2}\left(\left(1+\sqrt{2}\right)\left(3+2\sqrt{2}\right)^{n-1} - \left(\sqrt{2}-1\right)\left(3-2\sqrt{2}\right)^{n-1}\right) \\ &= N_n = R.H.S \end{aligned}$$

This proves 5; $2P_{n+1} * P_n + (P_{n+1})^2 - (P_n)^2 = N_n$

6 To Prove: We show that some algebraic expressions of consecutive terms of Pell sequence form a Pythagorean triplet.

Pythagorean Triplet$(a_n\,, b_n\,, h_n) = (2P_{n+1} * P_n\,, (P_{n+1})^2 - (P_n)^2\,, (P_{n+1})^2 + (P_n)^2)$
$\quad if\ n\ is\ an\ odd\ positive\ integer$
Pythagorean Triplet$(a_n\,, b_n\,, h_n) = ((P_{n+1})^2 - (P_n)^2\,, 2P_{n+1} * P_n\,, (P_{n+1})^2 + (P_n)^2)$
$\quad if\ n\ is\ an\ even\ positive\ integer$

In order to prove above relations we consider two cases for the integer **n,** to be either odd or even**,** and the general terms of NSW sequence.
We have already proved in the result **4**, listed above, that $(P_{n+1})^2 + (P_n)^2 = h_n\ where\ n \in N$

Case 1: Let n be an odd positive integer.

Consider $a_n = \frac{N_n - 1}{2} = \frac{1}{4}\left(\left(1+\sqrt{2}\right)\left(3+2\sqrt{2}\right)^{n-1} - \left(\sqrt{2}-1\right)\left(3-2\sqrt{2}\right)^{n-1} - 2\right)$

$$\begin{aligned} &= \frac{1}{4}\left(\left(1+\sqrt{2}\right)\left(1+\sqrt{2}\right)^{2n-2} - \left(\sqrt{2}-1\right)\left(1-\sqrt{2}\right)^{2n-2} - 2(-1)^{n-1}\right) \\ &= \frac{1}{4}\left(\left(1+\sqrt{2}\right)^{2n-1} + \left(1-\sqrt{2}\right)^{2n-1} - 2(-1)^{n-1}\right) \\ &= 2P_{n+1} * P_n \end{aligned}$$

This proves $a_n = 2P_{n+1} * P_n$ ….. (6.1)

Next Consider $b_n = \frac{N_n + 1}{2} = \frac{1}{4}\left(\left(1+\sqrt{2}\right)\left(3+2\sqrt{2}\right)^{n-1} - \left(\sqrt{2}-1\right)\left(3-2\sqrt{2}\right)^{n-1} + 2\right)$

$$\begin{aligned} &= \frac{1}{4}\left(\left(1+\sqrt{2}\right)\left(3+2\sqrt{2}\right)^{n-1} - \left(\sqrt{2}-1\right)\left(3-2\sqrt{2}\right)^{n-1} + 2(-1)^{n-1}\right) \\ &= \frac{1}{4}\left(\left(1+\sqrt{2}\right)\left(3+2\sqrt{2}\right)^{n-1} + \left(1-\sqrt{2}\right)\left(3-2\sqrt{2}\right)^{n-1} + 2(-1)^{n-1}\right) \\ &= (P_{n+1})^2 - (P_n)^2 \end{aligned}$$

This proves $b_n = (P_{n+1})^2 - (P_n)^2$ ….. (6.2)

From (6.1) and (6.2) we prove Pell equivalence for the Pythagorean triplet
$(a_n, b_n\,, h_n) = (2P_{n+1} * P_n\,, (P_{n+1})^2 - (P_n)^2\,, (P_{n+1})^2 + (P_n)^2)$ for n a positive odd integer.

Case 2: Let n be an even positive integer

Consider $a_n = \frac{N_n - 1}{2} = \frac{1}{4}\left(\left(1+\sqrt{2}\right)\left(3+2\sqrt{2}\right)^{n-1} - \left(\sqrt{2}-1\right)\left(3-2\sqrt{2}\right)^{n-1} - 2\right)$

$$\begin{aligned} &= \frac{1}{4}\left(\left(1+\sqrt{2}\right)\left(3+2\sqrt{2}\right)^{n-1} + \left(1-\sqrt{2}\right)\left(3-2\sqrt{2}\right)^{n-1} + 2(-1)^{n-1}\right) \\ &= (P_{n+1})^2 - (P_n)^2 \end{aligned}$$

This proves $a_n = (P_{n+1})^2 - (P_n)^2$ ….. (6.3)

Next Consider $b_n = \frac{N_n + 1}{2} = \frac{1}{4}\left(\left(1+\sqrt{2}\right)\left(3+2\sqrt{2}\right)^{n-1} - \left(\sqrt{2}-1\right)\left(3-2\sqrt{2}\right)^{n-1} + 2\right)$

$$= \frac{1}{4}\left(\left(1+\sqrt{2}\right)\left(1+\sqrt{2}\right)^{2n-2} - \left(\sqrt{2}-1\right)\left(1-\sqrt{2}\right)^{2n-2} - 2(-1)^{n-1}\right)$$

$$= \frac{1}{4}\left(\left(1+\sqrt{2}\right)^{2n-1} + \left(1-\sqrt{2}\right)^{2n-1} - 2(-1)^{n-1}\right)$$
$$= 2P_{n+1} * P_n$$

This proves $b_n = 2P_{n+1} * P_n$ ….. (6.4)

From (6.3) and (6.4) we prove Pell equivalence for the Pythagorean triplet
$(a_n, b_n, h_n) = ((P_{n+1})^2 - (P_n)^2, 2P_{n+1} * P_n, (P_{n+1})^2 + (P_n)^2)$ if n is positive even integer.

4. Derivation of a new sequence, NPJ, and its salient features

Consider the equation representing the interrelationship between Jha sequence and Pell sequence.

i.e. $J_n = P_n(P_n + P_{n-1}) \Rightarrow {P_n}^2 + P_nP_{n-1} - J_n = 0$, which is a quadratic equation and root are given by,

$$\frac{-P_{n-1} \pm \sqrt{{P_{n-1}}^2 + 4\,J_n}}{2}$$

Table 4: On Substituting Different Values of n We Get,

n	**First Root**	**Second Root (t_n)**
1	1	-1
2	2	-3
3	5	-7
4	12	-17
5	29	-41

Note that each term $(t_n, n = 1,2,3 \ldots)$ in the third column satisfies the recurrence relation

$$t_{n+2} = 2t_{n+1} + t_n\,, where\ n \in N\ with\ t_1 = -1\ and\ t_2 = -3 \qquad \text{….. (20)}$$

This sequence is called NPJ sequence and we use a notation NPJ for it.
Some terms of the sequence are -1, -3, -7, -17, -41, -99 … …... (21)
Using the recurrence relation we derive its general form,

$(NPJ)_n = \frac{(-1)}{2}\left[(1+\sqrt{2})^n + (1-\sqrt{2})^n\right]\ where\ n \in N\ and\ (NPJ)_n$ is the n^{th} terms of NPJ sequence. ….. (22)

The generating function is given by,

$$NPJ(x) = \frac{1+x}{x^2+2x-1} \qquad \text{….. (23)}$$

5. Sum of first n terms of Pell sequence

Before we begin with this section, we shall divide the set of natural numbers N in three mutually disjoint infinite sets as follows.

Define $N = A \cup B \cup C$

Where $A = \{n_1 | n_1 = 2n_j - 1, n_j \in N\}$
$B = \{n_2 | n_2 = 2(2n_j - 1), n_j \in N\}$
$C = \{n_3 | n_3 = 4n_j, n_j \in N\}$

To elaborate distinctly, we have $A = \{1,3,5,7 \ldots\}, B = \{2,6,10,14 \ldots\}, C = \{4,8,12,16 \ldots\}$
Clearly, $A, B\ and\ C$ are, in this case, infinite subsets of N such that$A \cap B \cap C = \emptyset$.
For any choice of $n \in N$ we can, with appropriate value of $n_j \in N$, immediately decide the set in which it falls.
It should be noted that the number n will lie in exactly one of the above subsets of N that is either in A or in B or in C.
We take up the following two examples.

E.g. for $n = 15$ we have $n_j = 8\ and\ so\ n \in A$
Also for $n = 20$ we have $n_j = 5\ and\ so\ n \in C$

Now we are going to develop a general formula to find the sum of first n terms of Pell sequence where $n \in N$. We denote this sum by S_n where n is a member of any one of the three infinite sets mentioned earlier.
We take up the following three cases:

Case 1: If $n \in A$ i.e. n is an odd positive integer of the form $2n_j - 1;\ n_j \in N$ then

$$S_n = S_{n_1} = \frac{N_{n_j}-1}{2}\ where\ N_{n_j} \text{ is the } n_j^{th} \text{ term of NSW sequence.} \qquad \text{….. (24)}$$

Example: Take $n = 7 = 2(4) - 1 \Rightarrow n_j = 4$
Therefore $S_7 = \frac{N_4-1}{2} = \frac{239-1}{2} = 119 = \sum_{n=1}^{7} P_n$

Case 2: If $n \in B$ i.e. n is an even positive integer of the form $4n_j - 2;\ n_j \in N$ then

$$S_n = S_{n_2} = \left(N_{n_j}\right)^2 = \left(J_{n_j+1} + J_{n_j}\right)^2 \qquad \text{….. (25)}$$

$where\ N_{n_j}$ is the n_j^{th} term of NSW sequence and J_{n_j} is the n_j^{th} term of Jha sequence

[Note that such positive naturals do not claim to possess a primitive Pythagorean triplet.]
Example: Take $n = 10 = 4(3) - 2 \Rightarrow n_j = 3$
Therefore $S_{10} = (N_3)^2 = 41^2 = 1681 = \sum_{n=1}^{10} P_n$
Also $S_{10} = (J_4 + J_3)^2 = (35 + 6)^2 = 1681 = \sum_{n=1}^{10} P_n$

Case 3: If $n \in C$ i.e. n is an even positive integer, a multiple of 4, of the form $4n_j;\ n_j \in N$ then

$$S_n = S_{n_3} = 8\left(J_{n_j+1}\right)^2\ where\ J_{n_j} \text{ is the } n_j^{th} \text{ term of Jha sequence} \qquad \text{….. (26)}$$

Example: Take $n = 8 = 4(2) \Rightarrow n_j = 2$
Therefore $S_8 = 8(J_{2+1})^2 = 8(J_3)^2 = 8(6)^2 = 288 = \sum_{n=1}^{8} P_n$

6. Conclusion

In the above note on sequences, our entire attention was focused on the deriving and discussing salient features of the three different sequences and right triangles of Fermat family. The prime focus then descended on to finding the sum of the first n number of terms, for any n$\in N$, of Pell sequence. It reflects a unique combination of the three sequences employed according to the situation that arises at a given point of time as on the selection of ‘n’ and hence ‘S_n’.

Annexure 1: In this section we derive the generating function of some sequences which are mentioned earlier.

1) Generating Function of Jha Sequence

Consider some terms of the sequence which are 0, 1, 6, 35, 204 … and the corresponding recurrence relation $J_{n+2} = 6J_{n+1} - J_n\ with\ J_1 = 0\ and\ J_2 = 1, n \in N$
Now we define power series,

$$\begin{aligned} J(x) &= 0 + x + 6x^2 + 35x^3 + \cdots \\ &= \sum_{n=1}^{\infty} J_n x^{n-1} \\ &= x + \sum_{n=3}^{\infty} J_n x^{n-1} \\ &= x + \sum_{n=1}^{\infty} J_{n+2} x^{n+1} \end{aligned}$$

Using the recurrence relation $J_{n+2} = 6J_{n+1} - J_n$

$$\begin{aligned} J(x) &= x + \sum_{n=1}^{\infty} [6J_{n+1} - J_n] x^{n+1} \\ &= x + 6x \sum_{n=1}^{\infty} J_{n+1} x^n - x^2 \sum_{n=1}^{\infty} J_n x^{n-1} \end{aligned}$$

Note that $\sum_{n=1}^{\infty} J_{n+1} x^n = x + 6x^2 + 35x^3 + \cdots = J(x)$

Therefore, $J(x) = x + 6x\,J(x) - x^2\,J(x)$
On simplifying we get,

$$J(x) = \frac{x}{1-6x+x^2}$$

This is a generating function of Jha Sequence.

2) Generating Function of NPJ Sequence
Consider some terms of the sequence which are $-1, -3, -7, -17$ ... and the corresponding recurrence relation, $t_{n+2} = 2t_{n+1} + t_n, with\ t_1 = -1\ and\ t_2 = -3$ where t_n represents the n^{th} term of NPJ sequence.
Now we define the power series,

$$\begin{aligned} t(x) &= -1 - 3x - 7x^2 - 17x^3 - \cdots \\ &= \sum_{n=1}^{\infty} t_n x^{n-1} \\ &= -1 - 3x + \sum_{n=3}^{\infty} t_n x^{n-1} \\ &= -1 - 3x + \sum_{n=1}^{\infty} t_{n+2} x^{n+1} \end{aligned}$$

Using the Recurrence relation $t_{n+2} = 2t_{n+1} + t_n$

$$\begin{aligned} t(x) &= -1 - 3x + \sum_{n=1}^{\infty} [2t_{n+1} + t_n] x^{n+1} \\ &= -1 - 3x + 2x \sum_{n=1}^{\infty} t_{n+1} x^n + x^2 \sum_{n=1}^{\infty} t_n x^{n-1} \end{aligned}$$

But $\sum_{n=1}^{\infty} t_{n+1} x^n = -3x - 7x^2 - 17x^3 + \cdots = t(x) + 1$
Therefore $t(x) = -1 - 3x + 2x\,(t(x) + 1) + x^2\,t(x)$
On simplifying we get,

$$t(x) = \frac{1+x}{x^2 + 2x - 1}$$

This is a generating function of NPJ sequence.

3) Generating Function of NSW Sequence
Consider some terms of the sequence which are $1, 7, 41, 239, \ldots$ and corresponding recurrence relation $N_{n+2} = 6N_{n+1} - N_n\ with\ N_1 = 1\ and\ N_2 = 7$
Now we define the Power series,

$$\begin{aligned} N(x) &= 1 + 7x + 41x^2 + \cdots \\ &= \sum_{n=1}^{\infty} N_n x^{n-1} \\ &= 1 + 7x + \sum_{n=3}^{\infty} N_n x^{n-1} \\ &= 1 + 7x + \sum_{n=1}^{\infty} N_{n+2} x^{n+1} \end{aligned}$$

Using the Recurrence relation $N_{n+2} = 6N_{n+1} - N_n$

$$N(x) = 1 + 7x + \sum_{n=1}^{\infty} [6N_{n+1} - N_n] x^{n+1}$$

$$= 1 + 7x + 6x\sum_{n=1}^{\infty} N_{n+1}x^n - x^2\sum_{n=1}^{\infty} N_n x^{n-1}$$

But $\sum_{n=1}^{\infty} N_{n+1}x^n = 7x + 41x^2 + 239x^3 + \cdots = N(x) - 1$
Therefore, $N(x) = 1 + 7x + 6x[N(x) - 1] - x^2\, N(x)$
On simplifying we get,

$$N(x) = \frac{1 + x}{1 - 6x + x^2}$$

4) Generating function of sequence of hypotenuse

Consider some terms of the sequence which are $1, 5, 29, 169, \ldots$ and corresponding recurrence relation $h_{n+2} = 6h_{n+1} - h_n \; with \; h_1 = 1 \; and \; h_2 = 5$
Now we define the power series,

$$\begin{aligned} h(x) &= 1 + 5x + 29x^2 + \cdots \\ &= \sum_{n=1}^{\infty} h_n x^{n-1} \\ &= 1 + 5x + \sum_{n=3}^{\infty} h_n x^{n-1} \\ &= 1 + 5x + \sum_{n=1}^{\infty} h_{n+2} x^{n+1} \end{aligned}$$

Using the Recurrence relation $h_{n+2} = 6h_{n+1} - h_n$

$$\begin{aligned} h(x) &= 1 + 5x + \sum_{n=1}^{\infty} [6h_{n+1} - h_n] x^{n+1} \\ &= 1 + 5x + 6x\sum_{n=1}^{\infty} h_{n+1}x^n - x^2\sum_{n=1}^{\infty} h_n x^{n-1} \end{aligned}$$

But $\sum_{n=1}^{\infty} h_{n+1}x^n = 5x + 29x^2 + \cdots = h(x) - 1$
Therefore, $h(x) = 1 + 5x + 6x[h(x) - 1] - x^2\, h(x)$
On simplifying we get,

$$h(x) = \frac{1 - x}{1 - 6x + x^2}$$

References

1. Horadam AF, Pell identities. The Fibonacci Quarterly. 1971; 9(3): 245-263.
2. Ahmet Dasdemir, On the Pell, Pell-Lucas and Modified Pell Numbers By Matrix Method, Applied Mathematical Science. 2011; 5(64):3173-3181.
3. Herbert s. Wilf, generating functionology, Second edition, 3-65.
4. Melham R, Sums involving Fibonacci and Pell Numbers, Portugaliae Mathematica. 1999; 56(3):309-317.
5. Mihir B, Trivedi, Pradeep J. Jha, From Pell to Pell$^+$ and CDS Sequences and Salient Features of Their Generator Matrices, IOSR Journal of mathematics. 2016; 12(4):05-10. (IOSR-JM).
6. http://mathworld.wolfram.com/NSWNumber.html