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Coding of right triangle and statistics of triplets

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Abstract
This note, according to its content in the mathematical proceedings has been classified in two sections. The first one tackles with general form of even and odd triples and their exhaustive properties corresponding right triangles. The second one corresponds to nature and number of primitive triplets their associated statistics followed by graphs taken on uniform interval of natural numbers. In short, the second part is all about statistics of triplets on a range of interval of positive integers. It also shows salient features with some open conjectures.

Keywords: primitive triplets, hypotenuse, jha sequence pell sequence, NSW * sequence

1. Introduction

In this note we have two different goals to elaborate fully with its inclination to mathematical properties.
The first section is devoted to, as said earlier, classical narration of triples and its characteristics. Instead of mentioning the three terms (a, b and h) of a right triangle, that is the value or the natural number associated with its shorter leg 'a' which is referred by 'odd' or 'even'. This 'odd' or 'even' value is referenced by odd or even natural number of the shorter leg. There are two different but complimentary stages to each other. Either you can specify any given positive integer which probably serves as a shorter leg or you can distinctly clarify that you need to find all related properties corresponding to n^{th} ordered even or odd triangle and that will be sufficient enough to give all geometric details concerning to that particular odd or even right triangle. The geometric properties refer to properties like the two other sides, area, inradius, and circumradius etc. These all characteristics are just related to the right triangle that is referenced by the primitive triplet only. As a special case we include Fermat family of right triangles. It is a family of right triangles such that the absolute difference of two shorter legs of any member triangle, i.e. $|b - a| = 1$ and their sum corresponds to a stipulated member of some known sequence; like 'Jha' sequence or an 'NSW' sequence.
In the second section we have, based on computer program, found all the primitive triplets for the possible integers that exist from 3 up to 5000. We have strictly followed the conditions defining a primitive triplet. In addition to this, corresponding statistics followed by graphs and possible deductions with conjectures have been also an important but serve as concluding remarks.

2 Primitive triplets and Salient Features:

A primitive Pythagorean triplet is a tuple like (a, b, c) where a, b, and c are some natural numbers that follow the following conditions.

(1) $a < b < c$ (2) $(a, b) = 1$ (3) $a^2 + b^2 = c^2$

The number 'a' is the shorter leg ('b' the next greater one; as a < b) and 'c' is called the hypotenuse.
Remaining within the norms of definitions we have some salient features described below.

(1) Under the prevalence of condition (2), we need to have either 'a' to be an Even/odd integer and simultaneously 'b' an odd/ even integer.
(2) The third condition claims that the hypotenuse must be an odd integer.
(3) Condition (1) a < b < c
Just to cite an example, we say that the triplet $(a, b, c) = (3, 4, 5)$ falls under the brackets of definitions of primitive triplets but $(a, b, c) = (4, 3, 5)$ does not.
Such points are clear on the basis of our definition.
(4) Natural numbers which are the members of the set $\{2(2n - 1) | n \in N\}$ cannot
Possess * any primitive triplet. [Proof is given at the end of this unit.]

(5) Given the value of ‘a’ only, we have structured our own results that will find the two quantities ‘b’ and ‘c’ defined as follows.

b = $\frac{a^2 - i^2}{2i}$, if ‘a’ is an odd integer so is ‘i’ also.

[If ‘a’ is even then ‘i’ is also even.]
In the same way we can write for ‘c’ also.

c = $\frac{a^2 + i^2}{2i}$, in this case also, if ‘a’ is an odd integer so is ‘i’ also.

[If ‘a’ is even then ‘i’ is also even.] In this way we have a primitive structure as

$$\left(a, b = \frac{a^2 - i^2}{2i}, c = \frac{a^2 + i^2}{2i}\right)$$

At this point, it is noteworthy that for an **odd** integer ‘a’ i = 1 assures the first primitive triplet and in the same way for an even value of ‘a’, i =2 assures to generate a primitive triplet. A Triplet is referred to as odd or even one depending upon the value of ‘a’ that is odd or even.

[***Theorem**: To show that the natural numbers like 2(2n – 1), n $\in$ N do not possess primitive Pythagorean triplets.]
Proof:

Let a = 2(2n – 1) for some n $\in$ N. ----- (1)

It means that the number is even.
As the derived results for finding the first primitive triplet is claimed for i=2, we proceed as follows.
By definition of a primitive triplet, (a, b) = 1 (i.e. GCD of a and b =1)

Now we have, b = $\frac{a^2 - 4}{4}$ is an odd integer only if ‘a’ is an even one.

In this case
$$b = \frac{\{2(2n-1)\}^2 - 4}{4}$$
$$= (2n-1)^2 - 1$$
$$= 4n(n+1)$$ (an even integer)

This implies that ‘a’ and ‘b’ both are even integer.

$\therefore$ (a, b) = 2 ----- (2)

Which contradicts primitiveness of the triplet.
Hence the proof.
We define primitive triplets and state all important salient features. We have proved some of them in the annexure. There are integers which possess more than 1 primitive triplets. There are integers which possess 2, 3, 4, 6, 7, 8… primitive triplets. We have found frequencies of such triplets for the numbers up to 5000.
Some important and eye-catching observations have been cited and attempts to establish them without loss of generality are pursued.
Graphical presentation of the above mentioned points have been done to elaborate various results on the given range (up to 5000).
At this stage it is noteworthy that some positive even integers of the form $2(2n-1)$; For $n \in N$ cannot possess any primitive triplet. (Proof shown in the above).
In addition to these, we turn our focus on the two important phases of content.

3 Coding of triangles and trigonometrical properties.
In this section we plan to assign a numerical code to each triangle. This coding depends on the odd or even numeric value of the

shorter leg ‘a’. If ‘a’ is an odd integer we call the corresponding right triangle as an odd right triangle and an even right triangle in the case when ‘a’ is an even integer.
At this stage we may like to recapitulate the results for finding the next leg (=b) and the corresponding value of hypotenuse (=c) of the triangle. The results are as under

- ‘a’ – an odd integer, $b = \frac{a^2 - i^2}{2i}$, $c = \frac{a^2 + i^2}{2i}$

  For i = 1, 9, 25, 49….; odd integer.
  [For sure, i = 1 gives the first primitive triplet. For other primitive triplet / triplets if exist, i must be a perfect square of an odd number]

- ‘a’—an even integer, $b = \frac{a^2 - i^2}{2i}$, $c = \frac{a^2 + i^2}{2i}$
  For i = 2, 8, 18, 36, 49……; an even integer.

  [For sure, i=2 gives the first primitive triplet, for other primitive triplet / triplets if exist. i must be of the form which equals to $2(n)^2$; where n is any positive integer]

  [Note: The above mentioned observational facts, at this stage get transformed to two corresponding conjectures with parallel statements in general.
  I: For the shorter leg [= a] of a right triangle being an odd integer, the next leg [or legs] other than the first primitive, if exist can be found for some i being odd positive square entries only.
  II: For the shorter leg [= a] of a right triangle being an even integer, the next leg [or legs] other than the first primitive, if exist can be found for some i being only for even multiple of odd positive square entries only.
  In addition to this we note (as proved above) that natural numbers of the form $(4n - 2) = 2(2n - 1)$ do not possess any primitive triplets. Now we initiate coding procedure.
  Now our discussion is divisioned for odd natural numbers (Part-I) and even natural numbers (Part-II) followed by some trigonometrical properties.

Part-I:
This fact relates to the first primitive triplet corresponding to the right triangle
We enlist some values.

n =	**1**	**2**	**3**	**4**	**…………**	**N**
2n +1 =	3	5	7	9	…………	2n+1

For this as done previously,

$$b = \frac{a^2 - 1}{2} = \frac{(2n+1)^2 - 1}{2} = 2n(n+1)$$
$$c = \frac{a^2 + 1}{2} = \frac{(2n+1)^2 + 1}{2} = 2n(n+1) + 1$$

i.e.; for n^{th} odd integer;

$$a = 2n + 1$$
$$b = 2n(n+1) \ \ and\, c = 2n(n+1) + 1$$

This helps derive some trigonometric results:

Perimeter $= 2s = a + b + c = 2(n+1)(2n+1)$
Area of the Right triangle $= \ n(n+1)(2n+1)$

Inradius of the triangle $= n$

Circumradius of the triangle = $\frac{2n(n+1)+1}{2} = n(n+1) + \frac{1}{2}$

All these results can be verified by putting integer values of n $\in N$
A few related terms are shown in the following table.

Table 1

N	2n+1 = a	b	c	2s=perimeter	area	R	r
1	3	4	5	12	6	2.5	0.5
2	5	12	13	30	30	6.5	1
3	7	24	25	56	84	12.5	1.5
4	9	40	41	90	180	20.5	2
5	11	60	61	132	330	30.5	2.5
6	13	84	85	182	546	42.5	3
7	15	112	113	240	840	56.5	3.5
8	17	144	145	306	1224	72.5	4
9	19	180	181	380	1710	90.5	4.5
10	21	220	221	462	2310	110.5	5
11	23	264	265	552	3036	132.5	5.5
12	25	312	313	650	3900	156.5	6

Part- II
In this section, as said earlier, we discuss about coding of even ('a' even) triplets.
Value of n = 1, 2, 3 …n
Corresponding 'a'= 4(n+1) = 8, 12, 16,…… 4(n+1)

Corresponding 'b' = $\frac{a^2-2^2}{4} = \frac{16(n+1)^2-4}{4} = 4(n+1)^2 - 1 = (2n+1)(2n+3)$

Corresponding 'c' = $\frac{a^2+2^2}{4} = \frac{16(n+1)^2+4}{4} = 4(n+1)^2 + 1$

From the above derivation we decide the followings.

1. Perimeter $= 2s = a + b + c = 4(n+1)\,(2n+3)$
2. Area of the triangle $= \Delta = \frac{1}{2} a \cdot b = 2(n+1)(2n+1)(2n+3)$
3. Inradius of triangle $= \frac{\Delta}{s} = (2n+1)$
4. Circumradius of the triangle $= \frac{abc}{4\Delta} = 2(n+1)^2 + \frac{1}{2}$

All these results can be verified by putting integer values of n $\in N$
A few related terms are shown in the following table.

Table 2

N	4(n+1)= a	B	c	2s=perimeter	Area	R	r
1	8	15	17	40	60	8.5	1.5
2	12	35	37	84	210	18.5	2.5
3	16	63	65	144	504	32.5	3.5
4	20	99	101	220	990	50.5	4.5
5	24	143	145	312	1716	72.5	5.5
6	28	195	197	420	2730	98.5	6.5
7	32	255	257	544	4080	128.5	7.5
8	36	323	325	684	5814	162.5	8.5
9	40	399	401	840	7980	200.5	9.5
10	44	483	485	1012	10626	242.5	10.5
11	48	575	577	1200	13800	288.5	11.5
12	52	675	677	1404	17550	338.5	12.5

Comment

The above mentioned points helps one derive trigonometrical results simply by giving the integer 'k' – i.e. 'kth' triangle;
Where k may be an odd or even integer. If k is an odd integer, then the formulae derived in terms of k (n=k) and same way we got for k even in part-II, will suffice enough information for the properties of triangle.

Part-III

In the final part of this section we, without sinking in rigorous mathematics, discuss about general nature of right triangles of Fermat family.

A right triangle is a member of Fermat family only if $|b-a|=1$ in the corresponding (primitive) triplet (a,b,c). Some of the triangles of this family are

	a_i	b_i	c_i
(1)	0	1	1
(2)	3	4	5
(3)	20	21	29
(4)	119	120	169
(5)	656	657	985…..

At this point, we refer to two standard sequences.

1. **NSW* Sequence**
   It is a sequence of positive integers having a well-defined recurrence relation.
   The sequence is $f_n = 1,7,41,239 \ldots.$

   With $f_1 = 1,\ f_2 = 7$ and $f_{n+2} = 6\ f_{n+1} - f_n$; For all $n \geq 1$ ---- (3)

   It is clear that; $f_n = a_n + b_n \quad for\ \forall\ n \epsilon N$ ---- (4)

   Where $a_n\ and\ b_n$ are the terms of Fermat triangle which satisfy $|b_n - a_n| = 1\ \forall\ n\epsilon N$

   To add to this, the hypotenuse of the Fermat triangles, when written in the form of an infinite sequence are 1, 5, 29, 169….. ---- (5)
   These terms can be, without any loss of generality, associated with the even ordered terms of widely known Pell Sequence.

2. **Pell Sequence**
   A few members of Pell sequence $P(n)$ are $0,1,2,5,12,29,70,169 \ldots.$

   With $P(1) = 0\ , P(2) = 1$

   And $P(n+2)\ =\ 2P(n+1) + P(n)$; for all $n \geq 1$ ---- (6)

This is the recurrence relation of it.

For this sequence each P(n+1) term for $n \in N$ is, without loss of generality, the corresponding term of sequence of hypotenuse in Fermat triangles.

3. **Jha Sequence**

This is an important sequence derived from the values of inradius of consecutive Fermat triangles. It has a wide application.A few terms of Jha sequence are as follows.

$$J_n: 0, 1, 6, 35, 204, \ldots$$

It has a recurrence relation; $J_{n+2} = 6J_{n+1} - J_n$ for $n \epsilon N$ with $J_1 = 0$, $J_2 = 1$.
Terms of this sequence can be used to establish mathematical relation with trigonometrical properties of Fermat triangle.

4 Statistics of triplets

In the above section we have seen all about Pythagorean triplets and with the same load we are inspired to perform basic statistics on the different data. It is clear that some of the observations mentioned below are based on the detailed study on the exhaustive list of all the triplets which range up to 5000.
We have seen that the pattern of locating primitive triplets are different for odd and even positive integers.
We display the facts regarding the observation on triplets.

- In the range [1,100], there are exactly **73** numbers which possess triplets, either 1, 2, or 3, with 'a' –the shorter leg remaining constant.

  Facts

  These facts are supported by theoretical deductions and actual work aided by 'c' program. We state some results and then represent all such in a tabular form and then graph them for easy interpretation.

- E.g.; for the shorter leg 'a' ϵ [1,100], only 73 number yield primitive Pythagorean triplets. These have be further classified as follows.

  For some a ϵ [1,100]
  Odd Primitive triplets- 49
  Even Primitive Triplets -24
  Total-73

- Some numbers of the form $\{8n + 12/n \epsilon N\} = \{20, 28, \ldots \ldots 100\}$ have '2' primitive triplets. [Also these numbers 48, 88, 96 have '2' Primitive triplets.]

- Some numbers of the form $\{6n + 27 = 3(2n + 9)/n \in N\} = \{33, 39, 45^*, 51, 99\}$ posses '2' primitive triples.
  [As a special case a=45 corresponds to only one primitive triplets]

- There are exactly 2 numbers in [1,100]; they are a=60 and a=84 which possess '3' primitive triplets. These numbers correspond to n=6 and n=9 in the above set {8n+12}.

Thus the statistics is
Total number forming primitive triplets=73
Out of which one triplet =48
Even numbers having two triplets =12
Odd numbers having two triplets =11
Two even numbers [60 and 84] possessing three triplets=2
In this way, the analysis has been extended for each group of 100 integer from 1 to 1000. The result has been tabulated as below.
Start with 1-100;

Table 3

Triplets	Count/Number
1	48
2	23
3	2
4	0
5	0
6	0
7	0
8	0
9	0

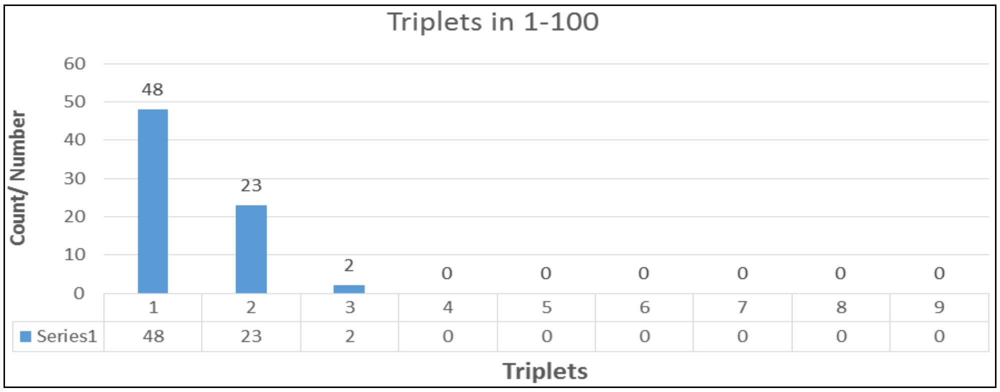


Fig 1

In the same way we have attempted analyze all numbers from 1 up to 1000 and has been tabulated as below.

Table 4

Triplets	1-100	101-200	201-300	301-400	401-500	501-600	601-700	701-800	801-900	901-1000	**Total**
1	48	33	29	25	24	22	23	23	21	21	**269**
2	23	33	32	35	35	36	32	33	32	34	**325**
3	2	9	9	10	10	8	12	7	12	10	**89**
4	0	0	5	5	5	9	7	11	9	9	**60**
5	0	0	0	0	0	0	0	0	0	0	**0**
6	0	0	0	0	1	0	0	1	1	1	**4**
7	0	0	0	0	0	0	1	0	0	0	**1**
8	0	0	0	0	0	0	0	0	0	0	**0**
9	0	0	0	0	0	0	0	0	0	0	**0**
Total	**73**	**75**	**75**	**75**	**75**	**75**	**75**	**75**	**75**	**75**	**748**

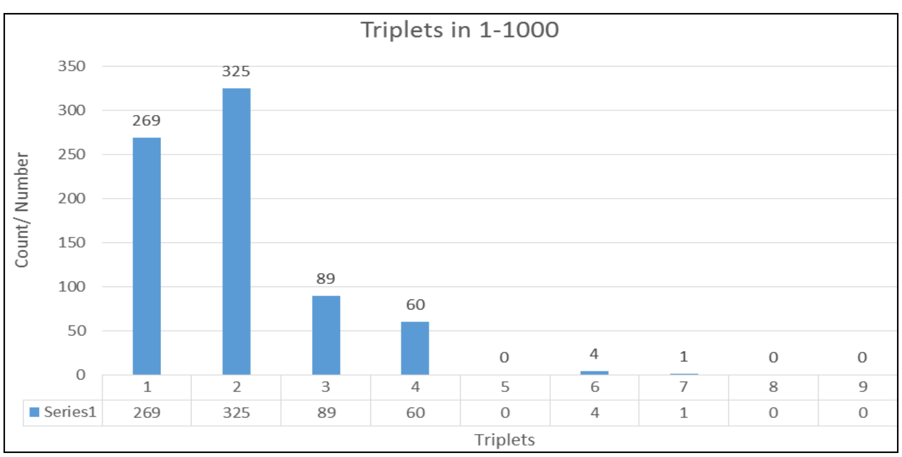


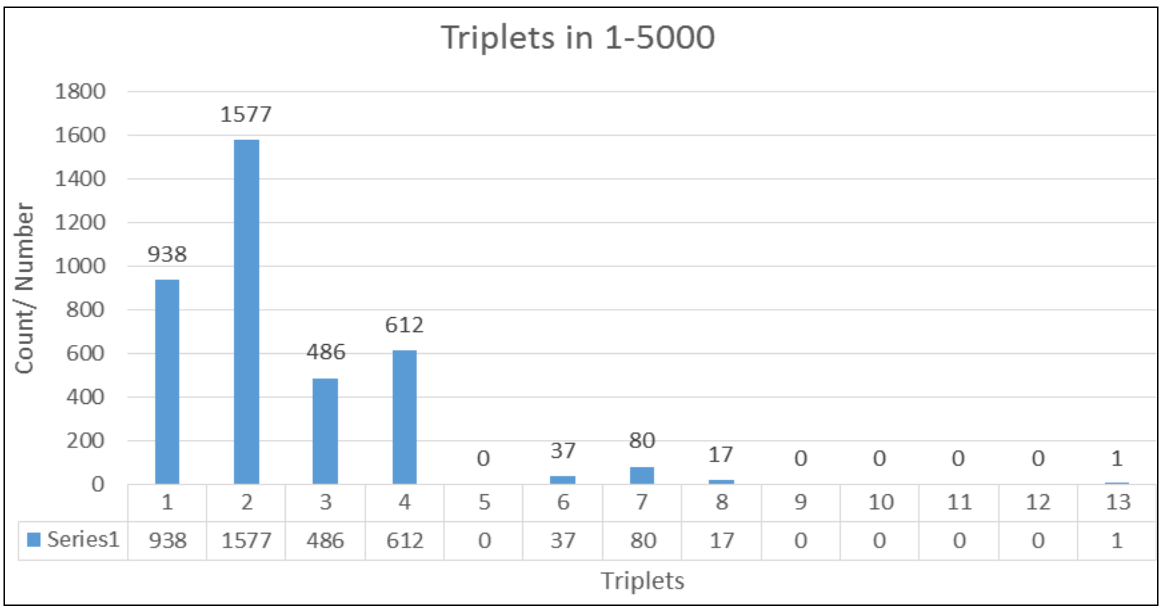
Fig 2

- In [1-1000]; there is no number which contains '5' primitive triplets.
  In the same way we have attempted analyze all numbers from 1 up to 5000 and has been tabulated as below.

Table 5

Triplets	1-1000	1001-2000	2001-3000	3001-4000	4001-5000	Total
1	269	188	166	162	153	938
2	325	325	315	311	301	1577
3	89	99	107	93	98	486
4	60	116	135	144	157	612
5	0	0	0	0	0	0
6	4	6	7	8	12	37
7	1	16	16	27	20	80
8	0	0	4	5	8	17
9	0	0	0	0	0	0
10	0	0	0	0	0	0
11	0	0	0	0	0	0
12	0	0	0	0	0	0
13	0	0	0	0	1	1
Total	748	750	750	750	750	3748

The above classification has been graphed as follows.



General Comment
Conjectures

- There is no number $a\epsilon[1,5000]$ for which there are 5 primitive Pythagorean triplets.
- There is no number $a\epsilon[1,5000]$ for which there are 9,10,11,12 primitive Pythagorean triplets.
- The number 4620 is the only number in [1, 5000] which possess 13 primitive Pythagorean triplets.

5. Conclusion

This note has shaded important outcomes in a way that has replaced any even or odd value of the shorter leg 'a' and its related properties in terms of its related natural number. This only is capable enough of elaborating all trigonometrical properties. The second part is to statistically represent the salient features of all possible triplets corresponding to natural numbers that possess triplets.

Vision

In addition to above facts, still there remains many astonishing facts about all facts pertaining to the hypotenuses of right triangles. Maximum attempts have been suggested in our next paper that describes nearly all characteristics of hypotenuses.

6. References

1. Ranna A, Trivedi, Shailesh A, Bhanotar, Dr. Pradeep J. Jha. : IOSR Journal of Mathematics (IOSR - JM) e - ISSN: 2278 - 5728, p - ISSN: 2319 - 765X. Volume 11, Issue 2 Ver. III (Mar - Apr. 2015), PP 54 – 63
2. Shailesh A. Bhanotar, Ranna A, Trivedi, Dr. Pradeep.J.Jha :International Journal of Applied Research 2015; 1(10): 820-826, IJAR 2015; 1(10): 820-826
3. Ranna A, Trivedi, Shailesh A. Bhanotar, Dr. Pradeep.J.Jha; The proceedings of the International Conference on Emerging Trends in Scientific Research : ICETSR- 2015,ISBN: 978-2-642-24819-9, C.U.SHAH University, (PP-223-224)
4. AA Hattangadi. Exploration in Mathematics, 3rd Edition India: University Press,2008.
5. Stifel, Michael, (1544), Arithmetica Integra(http:// mathdl.maa.org/math DL/46/?pa=content&sa=view document & nodeld=2591& body Id=3752
6. Ozanem, Jacques (1844)Science and natural philosophy: Dr. Huttons' Translation of Montucla's edition of Ozanam, revised by Edward Riddle, Thomas Tegg,London,Read online- Cornell University
7. Theoretical properties of the Pythagorean triples and connection to geometry (http://www.math.rutger.edu/ ~erowland/pythagoreantriples.html)
8. Generating Pythagorean Triples Using Arithmetic Progressions (http:// people.wcsu.edu/sandifere/Academics/2007Spring/Mat342/ PythagTrip02.pdf)