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Salient Features of 'c'- hypotenuse and related statistics

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Abstract
Contents of this paper deals with details- some of them proved and rest of them have been discussed- may be in the form of conjectures* but mostly true without loss of generality. Centuries have lapsed but, we think, some of the natural phenomena have remained partially known and rigorous version, may be all time, remains a mystery that need mathematical revolution from the known version. We have, to our best, tried to analyze and attempted to harmonize the data pertaining to the hypotenuses of right triangles and explaining some basic features have tried to represent in statistical formats finally presented its analysis in graphical form. [* conjectures like – There cannot exist a natural number that possesses Five (5) distinct primitive triplets. [This will call conjecture 1].
Comment: We have, using computer program, searched to look for such a number up to 10,000 natural numbers but have not found any natural number possessing the property said above. Ceaseless efforts are still continued in the direction and we expect some flashes of success. In fact, there are many structural suggestions, not yet that leads towards a right successful track.

Keywords: right triangle, hypotenuse, primitive triplet, family of right, triangles, semi prime

1. Introduction

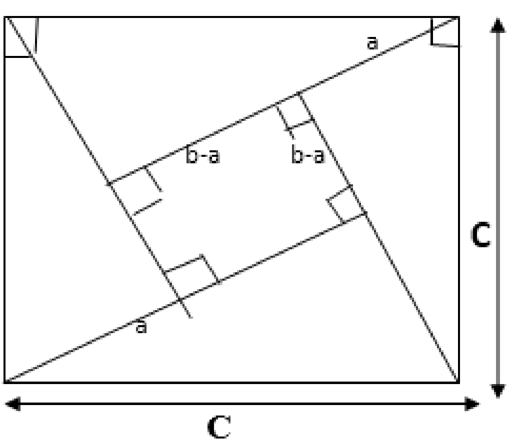
Right from the school days we have heard about right triangles and its classical properties that have already learnt, understood, and memorized by everyone who has completed school days. Yes, it is all about an all-time known personality –'Pythagoras (c.580B.C.- c. 500B.C.)'. He, as claimed by many literatures and published notes and many historians, has been a pioneer in establishing the facts that connects the three sides, say 'a'-the shorter leg, 'b'- the next one and 'c'- the side opposite to right angle called 'hypotenuse' in the form $a^2 + b^2 = c^2$.

There are many proofs [about 367 different proofs are known and published] given for the above fact corresponding to the sides of any right triangle. Although the credit to the search of Pythagorean Theorem often goes to Pythagoras but evidences surpass the claim to an Indian mathematician Baudhayana centuries earlier around 800 B.C.

Among the proofs of Pythagorean theorem, the smallest of all the proofs is given by an Indian mathematician Bhaskara $(1114.ca1185)$.

It shows the dissection of the square figure of side 'c' and compares the sum of areas of all right triangles within the square and the inner square with a side that equals (b-a). The proof is of one word -'Behold'.

Bhaskara's unique proof of Pythagoras assertion



Behold

Fig 1

Area of the four congruent triangles + area of the inscribed square

$4\left(\frac{1}{2}a \cdot b\right) + (b-a)^2 \Longrightarrow a^2 + b^2 = c^2$ = area of the square with side = c

At this point, we do not like to skip one additional fact in this direction to mention that the famous mathematician Euler (c. 300. B.C) asserted "Pythagorean theorem is reversible".

2 Pythagorean triplets and Primitive triplets

For the sake of convenience, we introduce notations as follows. The set of natural numbers N has been bifurcated as two infinite sets N_1 – a set of all odd integers and N_2 –a set of all even integers.

Clearly, $N = N_1 \cup N_2$ and $N_1 \cap N_2 = \phi$.

In connection to proceeding of this concept, many times we refer to Pythagorean triplets and Primitive Triplet.

2.1 Definition: Pythagorean triplets

A triplet (a, b, c), with a, b, and c some natural numbers is said to be a Pythagorean triplet, if it satisfies the condition: $a^2 + b^2 = c^2$

An infinite set of all such triples form a family of Pythagoras triplets is denoted by the notation 'P'. Though there are, according to hidden characteristics in the natural numbers **a, b,** and **c,** two more sub-families which are mentioned below.

2.2 Definition: Primitive triplets

A triplet (a, b, c), with a, b, and c some natural numbers, which satisfies the following properties, is known as primitive triplet / primitive Pythagorean triplet:

(1) G.C.D. of a and $b = 1$, Symbolically (a, b) = 1

(2) $a < b$

(3) $a^2 + b^2 = c^2$.

These properties help differentiate a primitive triplet from its multiples. There are many ways to construct Pythagorean triplets. Many mathematician and experts have found and divulged different ways to express their thought about triplets. We suggest one more of our own.

2.3 For odd integers

Let $a \epsilon N_1 - \{1\}$ be a given positive integer, then for some $i \epsilon N_1$, we have integers a, $b = \frac{a^2 - i^2}{2i}$, $c = \frac{a^2 + i^2}{2i}$, which satisfies $a^2 + b^2 = c^2$, and call (a, b, c) to form a Pythagorean triplet. [Note that 'c' is called the hypotenuse where c = b + i. We call a triplet an odd triplet if the smallest of all three integers. $i.e.$ a is an odd integer.

This calls for $\frac{a^2 + i^2}{2i} > \frac{a^2 - i^2}{2i} > a$.

Comment:

1 For odd integers i, $b = \frac{a^2 - i^2}{2i}$ is always an even integer (As a is an odd integer other than1)-

For example:

For $a = 3$, the integer $b = \frac{a^2 - i^2}{2i}$ $for\ i = 1, is\ 4$ and the next the highest in the sequence, we call it hypotenuse, $c = \frac{a^2 + i^2}{2i} = 5. i.e. 3^2 + 4^2 = 5^2$ is an odd Pythagorean Primitive triplet.

e.g., (3, 4,5), (5,12,17) … are an odd PPT.

2.4 For Even integers

There are some points of clarifications when the first integer of the primitive triplet i.e. a is an even integer. We will take up all such diversifications before they hit us on the way. Let $a \epsilon N_2 - \{2\}$. This means that for some a, an even integer, if we find $i \epsilon N_2$ so that the triplet of positive integers a, $b = \frac{a^2 - i^2}{2i}$, $and\ c = \frac{a^2 + i^2}{2i}$, where 'i' is an even integer, satisfy Pythagorean condition. $i.e.\ a^2 + b^2 = c^2$. (Note that $c = b + i$)

For example:

For $a = 8$ and $i = 2$, we have $b = \frac{a^2 - i^2}{2i}$ $\therefore b = \frac{8^2 - 2^2}{2(2)} = 15$ and $c = b + 2 = 15 + 2 = 17$.

The first even primitive Pythagorean triplet is (8, 15, 17).

1For $a = 12$ and $i = 2$, we have, $b = \frac{a^2 - i^2}{2i}$

$\therefore b = \frac{12^2 - 2^2}{2(2)} = 35$, and $c = b + 2 = 35 + 2 = 37$. Thus, we get the next even Pythagorean triplet (12, 35, 37).

To add to this, we have some views that help develop detail regarding Pythagorean triplets. In the primitive triplets with a, $b = \frac{a^2 - i^2}{2i}$, $c = \frac{a^2 + i^2}{2i}$ with $i \epsilon N_1\ or\ N_2$ as the case may be for $a \epsilon N_1\ or\ N_2$.

We have to our notes some known and obvious results.

1. From the properties inherent in the primitive Pythagorean triplets we deduce that if a is odd or even integer so is b even or odd integer and as a result the hypotenuse c is always an odd integer. Also as shown above, for the class of primitive triplets, $(a, b) = 1$ with $a < b$.
2. In most of the cases, at least one of the integers a, b and c is prime or divisible by 5.
3. A member integer of the set $\{1, 4\} \cup \{2(2n-1) | n \epsilon N\}$ cannot possess primitive Pythagorean triplet.
   [***Theorem**: To show that the natural numbers like 2(2n – 1), n∈ N do not possess primitive Pythagorean triplets.

Proof:

Let a = 2(2n – 1) is even for all n ∈ N. ----(1)

As the derived results for finding the first primitive triplet is claimed for i=2, we proceed as follows.

By definition of a primitive triplet, (a, b) = 1

Now we have, b = $\frac{a^2 - 4}{4}$ is an odd integer only if 'a' is an even one.

In this case $b = \frac{\{2(2n-1)\}^2 - 4}{4} = (2n-1)^2 - 1 = 4n(n+1)$ (an even integer)

This implies that ‘a’ and ‘b’ both are even integer.

∴ (a, b) = 2 ----- (2)

This result contradicts primitiveness of the triplet.
Hence the proof]

2.5 Different Families

Let us denote the universal set of triples of the form (a, b, h) by the symbol P. $P = \{(a,b,c)|\ a,b,c \in N, a^2 + b^2 = c^2\}$

1. **Plato Family P_1:** We define, the infinite set P_1 known as Plato Family of Pythagorean Triplets if for the given triplet (a, b, c), we have |c- b| = 1
$$P_1 = \{(a,b,c) \mid a,b,c \in N, a < b < c, (a,b) = 1, |c-b| = 1\}$$
Triplets like, $(3,4,5),(7,24,25) \in P_1$. Triplets are odd primitive triplets. This class is referred as Plato family.
2. **Pythagorean Sub-family P_2:** We define the infinite set P_2 known as Sub-family of Pythagorean set of triplet if for the given triplet (a, b, c), we have |c- b| = 2
$$P_2 = \{(a,b,c) \mid a,b,c \in N, a < b < c, (a,b) = 1, |c-b| = 2\}$$
Triplets like, $(8,15,17),(12,35,37) \in P_2$ are even primitive triplets. This class is referred as Pythagorean Sub-family.
3. **Fermat FamilyP_3**: We define, the infinite set P_3 known as Fermat Family of Pythagorean Triplets if for the given triplet (a, b, c), we have |a- b| = 1
$$P_3 = \{(a,b,c) \mid a,b,c \in N, a < b < c, (a,b) = 1, |b-a| = 1\}$$
Triplets like, $(3,4,5),(20,21,29) \in P_3$. Triplets of this class are always primitive triplets.
[*At this stage it is note- worthy that the set, $P' = P - \{P_1 \cup P_2 \cup P_3\}$, is an infinite set. Triplets like $(20,45,53),(39,80,89)$,…etc. are members of the infinite set P'.
*Also we note the fact that all the three families share a common triplet (3, 4, 5).]

2.6 Some Salient features of hypotenuse ‘c’:

Now we are equipped with focusing on the theme of this paper. Some of the dominating properties are listed below. A few of them are conjectures that have been verified up to 10,000 and found true but one can never claim generality.

1. c is always odd. [An obvious result for either exclusively ‘a’ or ‘b’ is odd or even [hence their squares] and hence the number ‘c’.
2. Hypotenuse **‘c’** is always an integer of the form 4n+5 for any natural number ‘n’. [It is important to note that the converse of this statement is not true in general. E.G. for n = 1, the number 4n + 5 = 9 which is not a hypotenuse for any triplet.]
3. The hypotenuse **‘c’** is widely classified into four groups[may not be mutually exclusive]
4. (1) A prime number (2) A square of an integer (3) Semi prime (4) Divisible by 5
[***1** Some square numbers like 25, 169, 289, 625, 841, etc. are hypotenuses corresponding to some ‘a’ and ‘b’.
The triplet (7, 24, 25) for which $7^2 + 24^2 = 25^2 = (5)^4$
The triplet (336, 527, 625) for which $336^2 + 527^2 = 625^2 = 5^8$
***2** In some cases the hypotenuse is a product of two primes (semi prime) E.G. In the triplet (21, 220, 221), the hypotenuse = 221 = 13 x 17, which is a product of two primes and no one of the prime is divisible by 5.
***3** It is an eye-catching fact that there are some hypotenuse = ‘c’ which correspond to *more than one primitive triplets*. The facts to an extent are described as follows.]

Number	Triplets
65	**2** -- (16, 63, 65), (33, 56, 65) [Like-wise 85, 145, 185………possess such characteristics.]
1885	**3** -- (427, 1835, 1885), (516, 1813, 1885), (924, 1643, 1885)
1105	**4** –(47, 1104, 1105), (264, 1073, 1105), (576, 943, 1105), (744, 817, 1105)

The fact shown above has been tabulated in the table -1 below.

[**Conjecture:** Those hypotenuses = **‘c’** which correspond to two or more primitive triplets are either ‘semi primes or divisible by 5.]

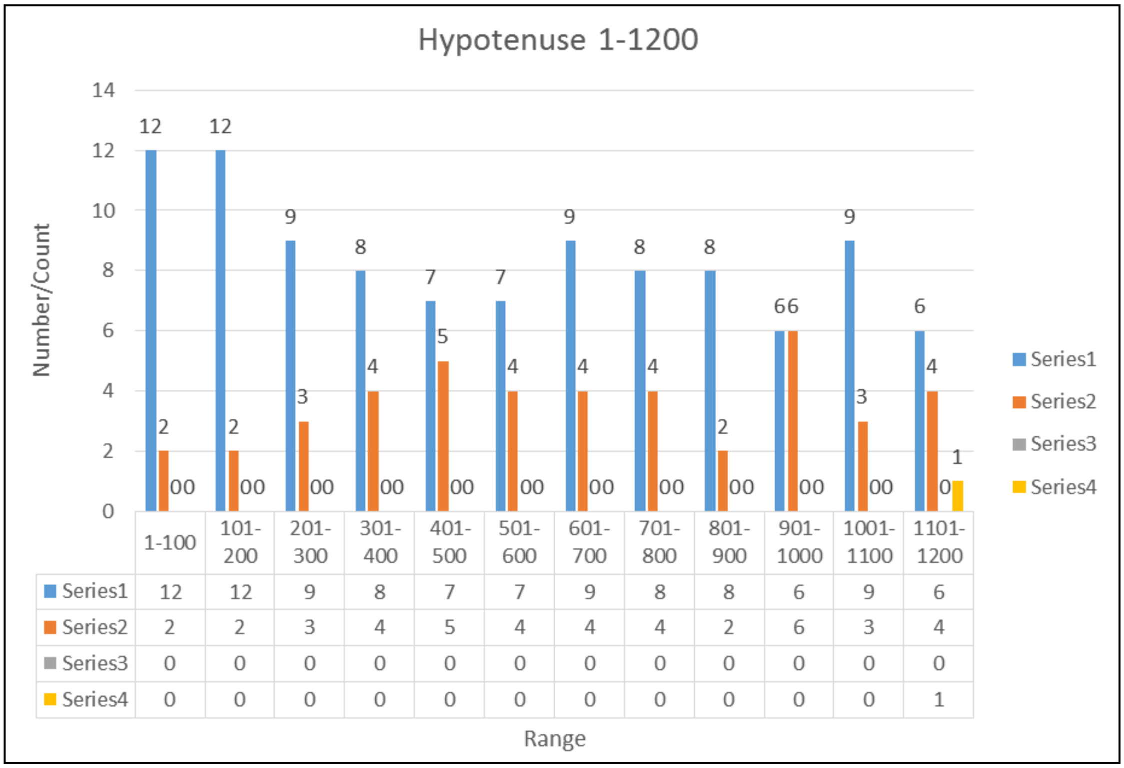
[This we call conjecture 2]

3. Total counts for Hypotenuse between 1 and 1200

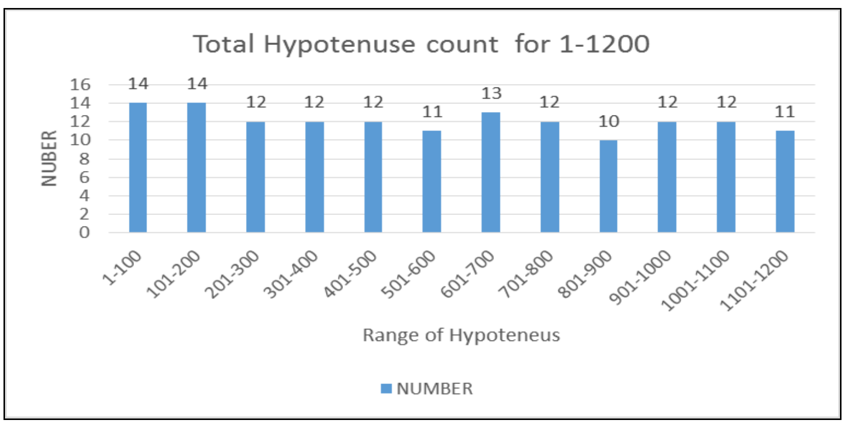
Now as we initiated in the introductory part, it is a time to close up with statistical part which begins with tabulation. This table sub groups the integers from 1 to 1200 indicating the total numbers taking part that represents the hypotenuses. It has been further (column-wise sub-divided into number of triplets)

Table 1: The tabulated information is graphically depicted below.

No. of Triplets	1 - 100	101-200	201-300	301-400	401-500	501-600	601-700	701-800	801-900	901-1000	1001-1100	1100-1200	Total
1	12	12	9	8	7	7	9	8	8	6	9	6	101
2	2	2	3	4	5	4	4	4	2	6	3	4	43
3	0	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0	0	1	1
Total	14	14	12	12	12	11	13	12	10	12	12	11	145



Graph 1: The following graph represents the same situation but it is in additive form that serves the purpose of general



Graph 2

A little forward towards interpretation is as follows.

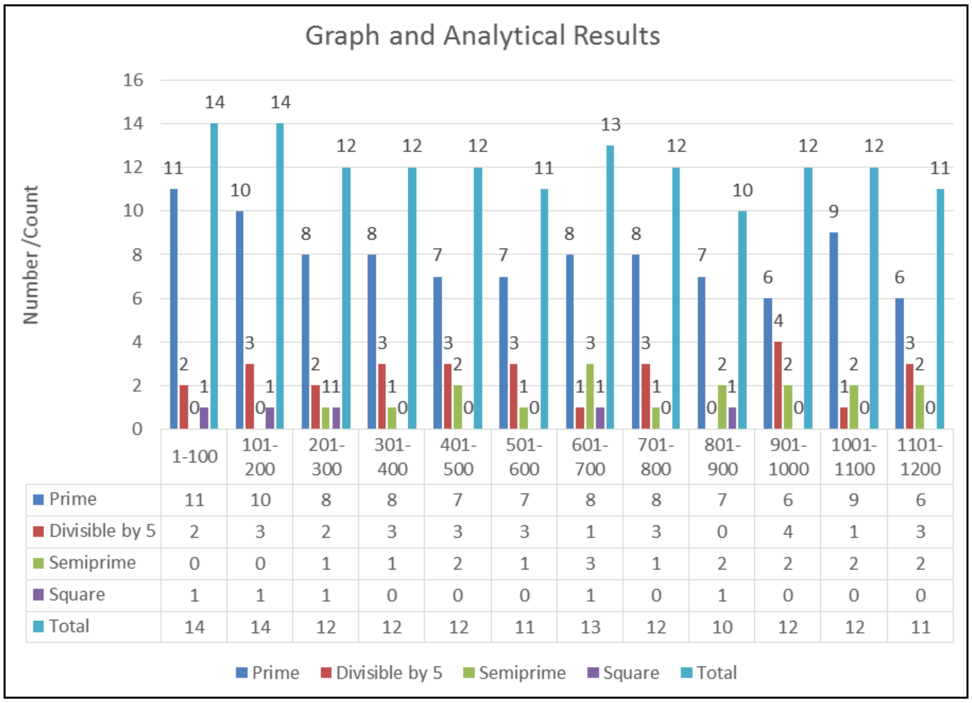
[Group	**Primes**	**Divisible by 5**	**Square Numbers**	**Semi Primes**	**Total**
1 --- 100	11 *	4 **	1***	2	14
[* Primes	5 13 17	29 37 41	53 61 73	89 97	= 11
** Divisible by 5	5 25	65 85			= 4
***Square number	25				= 1
****Semi Prime	65 85				= 2
Total:					= 14 #

The bold typed integers repeat twice and hence sharing part in two groups. If we count them once only then there are exactly 14 positive integers up to 100 which also contribute to being hypotenuse. Other details of different groups follow parallel]

Table 2: The following statistical table helps understand classification of hypotenuse up to 1200.

Range	Prime	Divisible by 5	Semi prime	Square	Total
1-100	11*	2(65,85)	-	1(25)	14
101-200	10	3(125,145,185)	-	1(169)	14
201-300	8	2(205,265)	1(221)	1(289)	12
301-400	8	3(305,325,365)	1(337)	-	12
401-500	7	3(425,445,485)	2(481,493)	-	12
501-600	7	3(505,545,565)	1(533)	-	11
601-700	8*	1(685)	3(629,689,697)	1(625)	13
701-800	8	3(725,745,785)	1(793)	-	12
801-900	7*	-	2(845,865)	1(841)	10
901-1000	6	4(905,925,965,985)	2(901,949)	-	12
1001-1100	9	1(1025)	2(1037,1073)	-	12
1101-1200	6	3(1105,1145,1165)	2(1157,1189)	-	11

Now we show a bar-chart of distribution of integers in interval of 100 integers versus its classification and it is followed by the outcome shown in tabular form.



Graph 3

This analysis boils down to one more conjecture [conjecture-3]. All the integers possessing two or more primitive triplets have at least one odd triplet and the rests are even triplets but contrary to this observation the integer 325 has only two even triplets. Are there any more hypotenuses with this property?

(1) a = 36, b= 323, c = 325 (2) a = 204, b = 253, c = 325

[Clearly a = 36 and a = 204 both are even integers that relate to c = 325.]

Conclusion

This inspiring paper giving rigorous analytical details about hypotenuse is highly informative and statistical flair has added to its completeness. In addition to this, some new flashes of assertion a some conjectures have truly made our attempts useful to aspirant mathematicians.

Vision

It remains open to prove conjectures true otherwise is probably our next goal but wide open to mathematical minded to pursue in the direction.

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