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Application of dziok-srivastava operator to various subclasses of analytic multivalent functions

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Abstract
In the present paper various new subclasses of analytic multivalent functions are introduced by applying Dziok-Srivatava operator have been introduced. Many interesting applications involving these and other families of integral operators are also discussed. Our results provide generalizations of inclusion properties established recently by Choi, Saigo and Srivastava (2002) and others have also been discussed.

Keywords: analytic functions, univalent and multivalent functions, starlike functions, convex functions, differential subordination, hadamard product (or convolution), integral operators

Introduction

Let A_p denote the class of functions f (z) *normalized* by

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n, \quad (p \in \mathbf{N}) \qquad \ldots(1.1)$$

which are analytic in the open unit disk U = {z: z ∈ C and | z | < 1 }. Also let $S_p^*(\alpha)$ and $K_p(\alpha)$ denote, respectively, the subclasses of A_p consisting of p-valent functions which are *starlike* and *convex* of order α in U with $0 \le \alpha < p$. In particular $S_p^*(0) = S_p^*$ and $K_p(0) = K_p$ are the well-known subclasses of p-valent *starlike* and p-valent convex functions in U, respectively.

Given two functions f and g, which are analytic in U with f (0) = g (0), the function f is said to be *subordinate* to g in U if there exists a function w, analytic in U, such that

$$w(0) = 0, \; |w(z)| < 1 \; (z \in U), \text{ and } f(z) = g(w(z)) \; (z \in U).$$

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We denote this subordination by

$$f(z) \prec g(z) \text{ in } U$$

We also observe that

$$f(z) \prec g(z) \text{ in } U$$

iff $\quad f(0) = g(0)$ and $f(U) \subset g(U)$

whenever g is *univalent* in U.

Let M be the class of analytic functions ϕ (z) in U normalized by ϕ (0) = 1, and let H be the subclass of M consisting of those functions ϕ which are univalent in U and for which ϕ (U) is convex and R {ϕ(z)} > 0 (z ∈ U).

We define the following subclasses $s_p^*(\phi)$ and $k_p(\phi)$ for ϕ ∈ H by

$$s_p^*(\phi) = \left\{ f : f \in A_p \text{ and } \frac{z f'(z)}{f(z)} \prec p\,\phi(z) \text{ in } U \right\}$$

$$k_p(\phi) = \left\{ f : f \in A_p \text{ and } 1 + \frac{z f''(z)}{f'(z)} \prec p\,\phi(z) \text{ in } U \right\},$$

Obviously

1. $s_p^*\left(\frac{1+z}{1-z}\right)=S_p^*$,
2. $k_p\left(\frac{1+z}{1-z}\right)=K_p$
3. $s_p^*\left(\frac{1+Az}{1+Bz}\right)=S_p^*[A,B]\ \ (-1\le B<A\le 1)$
4. $k_p\left(\frac{1+Az}{1+Bz}\right)=K_p[A,B]\ \ (-1\le B<A\le 1)$.

We also have $S_p^*[1,-1]=S_p^*$ and $K_p[1,-1]=K_p$. For p = 1, the above reduced classes $S^*[A,B]$ and K [A,B] were investigated by Janowski [9] and Goel and Mehrok [6].

Clearly

$$f(z)\in k_p(\phi)\Leftrightarrow z f'(z)\in s_p^*(\phi)\ .$$

Further suppose that

$$h_p[(\alpha_q);(\beta_r);z]=z^p\ {}_qF_r(\alpha_1,...,\alpha_q;\beta_1,...,\beta_r;z)$$

$$=z^p+\sum_{n=p+1}^{\infty}B_p^{(\alpha_q),(\beta_r)}(n)\,z^n \qquad \text{…(1.2)}$$

$$(q\le r+1;\alpha_i\in R;\beta_j\in R\setminus Z_0^-;Z_0^-=\{0,-1,-2,...\};$$

$i = 1,\ldots, q;\ j = 1,\ldots,r;\ z\in U)$

where ${}_qF_r$ is the generalized hypergeometric function and

$$B_p^{(\alpha_q),(\beta_r)}(n)=\frac{(\alpha_1)_{n-p}(\alpha_2)_{n-p}...(\alpha_q)_{n-p}}{(\beta_1)_{n-p}(\beta_2)_{n-p}...(\beta_r)_{n-p}(n-p)!} \qquad \text{…(1.3)}$$

Corresponding to the function $h_p[(\alpha_q);(\beta_r);z]$, Dziok and Srivastava [4, p.3, Eq.(3)] introduced a linear operator $\boldsymbol{H_{p,\alpha_q,\beta_r}}$ defined by the convolution

$$H_{P,\alpha_q,\beta_r}\ f(z)=h_p[(\alpha_q);(\beta_r);z]*f(z) \qquad \text{…(1.4)}$$

or equivalently by

$$H_{P,\alpha_q,\beta_r}\ f(z)=z^p+\sum_{n=p+1}^{\infty}B_p^{(\alpha_q),(\beta_r)}(n)\,a_n z^n\ \ (z\in U). \qquad \text{…(1.5)}$$

Here $*$ stands for the convolution of two analytic multivalent functions f and g of the form

$$f(z)=z^p+\sum_{n=p+1}^{\infty}a_n z^n,\ g(z)=z^p+\sum_{n=p+1}^{\infty}b_n z^n\ (a_n,b_n\ge 0,\ p\in N)$$

and is defined by

$$(f*g)(z)=z^p+\sum_{n=p+1}^{\infty}a_n b_n z^n. \qquad \text{…(1.6)}$$

The linear operator $H_{P,\alpha_q,\beta_r}\ f(z)$ includes various other linear operators considered earlier by Hohlov [8], Carlson-Shaffer [2], Goyal and Bhagtani [7], Ruscheweyh [13] etc.

Next by using the operator H_{p,α_q,β_r} we introduce the following classes of analytic functions for $\phi \in H$; $f \in A_p$ $\alpha_q > -1$ and $\beta_r \geq 1$

$$s^*_{p,\alpha_q,\beta_r}(\phi) = \{f : f \in A_p \text{ and } H_{p,\alpha_q,\beta_r} f(z) \in s^*_p(\phi)\}$$

$$k_{p,\alpha_q,\beta_r}(\phi) = \{f : f \in A_p \text{ and } H_{p,\alpha_q,\beta_r} f(z) \in k_p(\phi)\}$$

We also note that

$$f(z) \in k_{p,\alpha_q,\beta_r}(\phi) \Leftrightarrow z f'(z) \in s^*_{p,\alpha_q,\beta_r}(\phi) \qquad \ldots(1.7)$$

In particular, we set

$$s^*_{p,r,s,t,2}\left(\frac{1+z}{1-z}\right) = S^*_{p,r,s,t},$$

$$s^*_{p,\lambda,\nu,\eta,\mu}\left(\frac{1+Az}{1+Bz}\right) = S^*_{p,\lambda,\nu,\eta,\mu}[A,B] \quad (-1 \leq B < A \leq 1),$$

$$k_{p,\lambda,\nu,\lambda,\mu}\left(\frac{1+Az}{1+Bz}\right) = K_{p,\lambda,\nu,\eta,\mu}[A,B] \quad (-1 \leq B < A \leq 1),$$

and for p = 1, we have

if r = s = n then $s^*_{n,2}\left(\frac{1+z}{1-z}\right) = S^*_n$,

if $\nu = \lambda$, then $s^*_{\lambda,\mu}\left(\frac{1+Az}{1+Bz}\right) = S^*_{\lambda,\mu}[A,B] \quad (-1 \leq B < A \leq 1)$

and $\quad k_{\lambda,\mu}\left(\frac{1+Az}{1+Bz}\right) = K_{\lambda,\mu}[A,B] \quad (-1 \leq B < A \leq 1)$.

For a function $f \in A_p$ given by (1.1) the generalized Bernardi-Libera-Livingston integral operator $\mathbf{J}_{p,\gamma}$ is defined by (cf. [1], [10] and [11])

$$\mathbf{J}_{p,\gamma}(f)(z) = \frac{\gamma+p}{z^\gamma}\int_0^z t^{\gamma-1} f(t)\,dt \qquad \ldots(1.8)$$

$$(f \in A_p; \gamma > -p;\ p \in N)$$

$$= z^p + \sum_{j=p+1}^{\infty} \frac{\gamma+p}{\gamma+j} a_j z^j = z^p + \sum_{j=p+1}^{\infty} \frac{(\gamma+p)_{j-p}}{(\gamma+p+1)_{j-p}} a_j z^j$$

Also

$$z(H_{p,\alpha_q,\beta_r}(J_{p,\gamma}(f)(z))' = (\gamma+p) H_{p,\alpha_q,\beta_r} f(z) - \gamma H_{p,\alpha_q,\beta_r}(J_{p,\gamma}(f)(z)) \qquad \ldots(1.9)$$

where the operator H_{p,α_q,β_r} is defined by (1.5).

The operator $\mathbf{J}_{p,\gamma}$ (for p = 1, $\gamma \in N$) was introduced by Bernardi [1]. In particular, the operator $\mathbf{J}_{1,1}$ was studied by Libera [10] and Livingston [12]. In fact the case p = 1 of the operator $\mathbf{J}_{p,\gamma}$ appeared in numerous earlier works (see, e.g. Srivastava and Owa [14, p.66,154, 181 and 338]).

2. Inclusion Properties Involving $\mathbf{J}_{p,\gamma}$

The following result will be required in our investigation:

Lemma (Eenigenburg *et al.* [5]). *Let h be convex univalent in U with h (0) = 1 and*

$$R\{\beta h(z) + \gamma\} > 0 \qquad (\beta, \gamma \in C)$$

If p (z) is analytic in U with p (0) = 1, then

$$p(z) + \frac{z\,p'(z)}{\beta\,p(z)+\gamma} \prec h(z) \text{ in U}$$

implies that p (z) $\prec$ *h (z) in U.*

The following theorems deal with the generalized Bernardi-Libera-Livingston integral operator $\mathbf{J}_{p,\gamma}$ (f) defined by (1.8).

Theorem 1. *Let* $\alpha_q > -1$ and $\beta_r \geq 1$. *If* $f(z) \in s^{*}_{p,\alpha_q,\beta_r}(\phi)\ \ (\phi \in H)$, *then* $J_{p,\gamma}(f)(z) \in s^{*}_{p,\alpha_q,\beta_r}(\phi \in H)$.

Proof. Let $f(z) \in s^{*}_{p,\alpha_q,\beta_r}(\phi)$ for $\phi \in H$ and set

$$\frac{z\,(H_{p,\alpha_q,\beta_r}\ J_{p,\gamma}(f)(z))'}{H_{p,\alpha_q,\beta_r}\ J_{p,\gamma}(f)(z)} = p\,\theta(z) \qquad \ldots(2.1)$$

where $\theta(z) = 1 + c_1 z + c_2 z^2 + \ldots$ is analytic in U and $\theta(z) \neq 0$ for all $z \in U$.

By applying (1.9) and (2.1), we have

$$\frac{(\gamma+p)\,H_{p,\alpha_q,\beta_r}\ f(z)}{H_{p,\alpha_q,\beta_r}J_{p,\gamma}\ f(z)} = p\,\theta(z) + \gamma \qquad \ldots(2.2)$$

Making use of the logarithmic differentiation on both sides in (2.2), we get

$$\frac{z\,(H_{p,\alpha_q,\beta_r}\ f(z))'}{H_{p,\alpha_q,\beta_r}\ f(z)} = p\,\theta(z) + \frac{p\,z\,\theta'(z)}{p\,\theta(z)+\gamma} \qquad \ldots(2.3)$$

Since $\gamma \geq 0$, $\phi(z) \in H$ and $f(z) \in s^{*}_{p,\alpha_q,\beta_r}(\phi)$ from (2.3),we obtain

$$R\{p\phi(z)+\gamma\} > 0 \text{ and } p\,\theta(z) + \frac{p\,z\,\theta'(z)}{p\,\theta(z)+\gamma} \prec p\,\phi(z) \quad (z \in U)$$

Hence by virtue of Lemma, we conclude that

$p\,\theta(z) \prec p\,\phi(z)$ in U,

which implies that

$$\mathbf{J}_{p,\gamma}(f)(z) \in s^{*}_{p,\alpha_q,\beta_r}(\phi) \quad (\phi \in H).$$

Theorem 2. *Let* $\alpha_q > -1$ and $\beta_r \geq 1$. *If* $f(z) \in k_{p,\alpha_q,\beta_r}(\phi)\ (\phi \in H)$, *then*

$$\mathbf{J}_{p,\gamma}(f)(z) \in k_{p,\alpha_q,\beta_r}(\phi) \quad (\phi \in H).$$

Proof. By applying Theorem 1, it follows that

$$f(z) \in k_{p,\alpha_q,\beta_r}(\phi) \Leftrightarrow z\,f'(z) \in s^{*}_{p,\alpha_q,\beta_r}(\phi)$$

$$\Rightarrow J_{p,\gamma}(z\,f'(z)) \in s^{*}_{p,\alpha_q,\beta_r}(\phi) \Leftrightarrow z\,(J_{p,\gamma}(f)(z))' \in s^{*}_{p,\alpha_q,\beta_r}(\phi)$$

$$\Leftrightarrow J_{p,\gamma}(f)(z) \in k_{p,\alpha_q,\beta_r}(\phi) \qquad (\phi \in H)$$

which proves Theorem 2.

Remark 1. By comparing the definitions (2.5) and (1.8) rather closely and applying (1.5) we have the following relationship

$$\mathbf{J}_{p,\gamma}(f)(z) = L_p(\gamma+p, 1+\gamma+p)\ f(z) = H_{p,\gamma+p,1;\gamma+p+1}\ f(z) \quad (\gamma > -p) \qquad \ldots(2.4)$$

among the Carlson-Shaffer linear operator L_p (a,d), the generalized integral operator H^{p,α_q,β_r} and the generalized Bernardi-Libera-Livingston integral operator $\mathbf{J}_{p,\gamma}$ ($\gamma > -p$). This relationship (2.4) can alternatively be invoked in order to derive the results of this section from those of the preceding section. The details involved in these alternative derivations are being omitted here

Also the Carlson-Shaffer linear operator L_p (a,d) is defined as

$$L_p(a,d)\; f(z) = z^p + \sum_{n=p+1}^{\infty} \frac{(a)_{n-p}}{(d)_{n-p}} a_n z^n \qquad \text{….(2.5)}$$

For p = 1, q = r+1, $\alpha_i = \beta_i$, i = 1,2,…r and $\alpha_{r+1} = 1$, the aforementioned results contained in our Theorems 1 and 2 reduce to results established earlier by Choi *et al.* [3].

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