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Approaches differ: Catalan numbers

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Abstract
It has been said and written, stuffing pages, about Catalan numbers but till now it has remained a center of attraction to cast one's own interest and study the facts with different perspectives. The claim has found, on the top of all, that discovery has been authored by Chinese mathematician [Mingantu – 1730] before Catalan was successful on attempts initiated by Leonhard Euler (probably critically reviewed by Goldbach) in the search of triangulation of a polygon. Hopefully lined in the same direction we also, in this note, have attempted new approaches establishing recurrence relations and linkage of Narayan numbers with Catalan numbers.

Keywords: catalan numbers, narayana numbers

1. Introduction

As said earlier, before Catalan worked on Euler's attempted work on triangulation of an n-gon, a recently appeared in lime light, a Chinese mathematician Mingantu, dated back in 1730, worked in some area (Series expansion of $\sin 2\alpha$ and $\sin 4\alpha$in terms of $\sin\alpha$). Probably begun with the problem of triangulation of an n-gon, Catalan numbers have appeared sequentially as a solution to many combinatorial problem in Mathematics. Once it has found a path of speedy movement, people working in different branch of Mathematics have been off and on keep target on pursuing to tend or to converge on the same route that might give some credit to contribution. As known, there are more than 70 application oriented interpretation to Catalan numbers. Our attempts, in this note are also from some view points which probably fill in gaps for beginners in this area. We have also jotted down our reflexes and put them shading our views. The note may not be fully exhaustive but it has given us satisfaction.

1.1 About Catalan Numbers

As said earlier, basically it is an astounding segment of combinatorics. Catalan numbers are the numbers showing the ways of triangulation of a (n+2)-gon for sequential values of $n \in N.$

Some terms of the Catalan sequence (C_n) are as follows.

C_n: 1, 2, 5, 14, 42, 132, 429,

[Comment: It is, for some mathematicians pursuing pedantic/ traditional approach, customary to denote the members of Catalan sequence in the following format.

C_n: 1, 1, 2, 5, 14, 42,

where we keep n $\in$ N $\cup$ {0}

In this case we shall write that C_0 =1, C_1 = 1, and so on. Both the results have valid justification in the sense of interpretation.]

It is an infinite sequence with more even numbers, as observed, than the odd numbers are.

For example, for $n = 3, C_3 = 5$ shows the possible ways in which a pentagon (3+2 =5) can be triangulated in which two sides are non-intersecting diagonals and a side of a pentagon as a base.

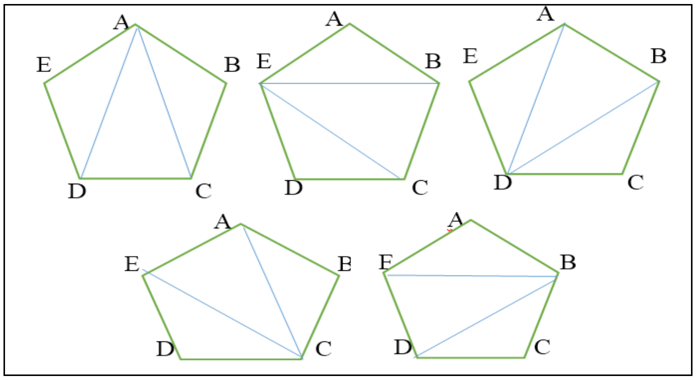

Fig 1

Above figures help us understand that the number $C_3 = 5$. Each one of the 5 sides participate in the process of triangulation which is the one subtending a triangle with opposite vertex. A little care should be taken for the triangulation process that no two ways of triangulations involve the same triangles as all that found in one but each pattern has the same **number** of triangles. This is what is meant by denoting $C_3 = 5$. That there are 5 different ways of triangulating a pentagon and each way has 3 triangles involved but in no case they all are same. The triangle contains exactly two non-intersecting diagonals and a side of the pentagon on hand.
In the same way for n= 4, in the process of triangulation of a hexagon (=4+2= 6), we have the fourth Catalan number i.e. $C_4 = 14$.
Thus logical formulation for the Catalan numbers is given by the result,

$C_n = \frac{1}{n+1}\binom{2n}{n}$ for n $\in$ N, resulting in to $C_n = 1, 2, 5, 14, \ldots$.. …. (1)

It can be easily derived that the above formula is mathematically equivalent to

$C_n = \frac{(2n)!}{(n+1)!(n)!}$ for n $\in$ N, resulting in to $C_n = 1, 2, 5, 14, \ldots$.. …. (2)

$C_n = \prod_{k=2}^{n}\left(\frac{n+k}{k}\right)$ [Feasible for n, and $k \geq 2$]

Also the Catalan numbers are solutions to Housedroff moments problem in the interval [0, 4].

1.2 Recurrence Relation and Important Deduction

In this unit we would like to express our work in three different sub units. These unit will help understand the different types of approaches. The first one is about the recurrence relation among the consecutive terms of the sequence.

1.2.1 Recurrence Relation

We, at this stage, using fundamental formula for n^{th} Catalan number derive a recurrence relation. This shall be useful in topics that follow.

We accept $C_n = \frac{1}{n+1}\binom{2n}{n}$

And establish that $C_{n+1} = \left(\frac{4n+2}{n+2}\right)C_n$

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{1}{n+1}\left(\frac{(2n)!}{n! * n!}\right)$$

We let $n+1$ for n and get

$$C_{n+1} = \frac{1}{n+2}\binom{2n+2}{n+1} = \frac{1}{n+2}\left(\frac{2(n+1)(2n+1)(2n)!}{(n+1)^2(n!)^2}\right)$$
$$= \frac{1}{n+2}\left(\frac{2(2n+1)}{1}\right)\left(\frac{1}{n+1} * \frac{(2n)!}{(n!)^2}\right) \therefore\ C_{n+1} = \frac{4n+2}{n+1}C_n \ldots \qquad \ldots\ldots. (3)$$

If we put $(n-1)$ for n then we have

$$C_n = \frac{4n-2}{n+1} C_{n-1} \quad for\ n \in N\ with\ C_0 = 1$$ ………. (4)

1.2.2 Euler's Formula from Catalan's General Formula:

In the following unit we derive Euler's formula for his efforts in search for triangulation to different polygons. As a result he deduced that the number of triangulation process yield the result; that the ways are $\frac{2*6*10*\ldots*(4n-2)}{(n+1)!}$

It is claimed that Catalan developed and extended his work and contributed a lot in the area of combinatorics.

We accept Catalan's general result;

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

and using principles of mathematical induction we can establish their equality.

$$C_n = \frac{2*6*10*\ldots*(4n-2)}{(n+1)!} \quad for\, n \geq 1$$ ………… (5)

2. Catalan Numbers: A step further

After some known results derived in our own format, we have two more units to be added to the note. The results shown are well tested without loss of generality.

2.1 Different Meaningful Results

In this unit we show different meaningful results pertaining to Catalan numbers; which are as follows.

2.1.1 Determinant form of Catalan Number

Catalan Numbers $C_n, n \in N$ can also be represented as determinant value of Matrix of order 2.
This Matrix is given by

$$M_n = \begin{bmatrix} \binom{2n-1}{n-1} & \binom{2n-1}{n-1} \\ \frac{2n}{n+1} & 2 \end{bmatrix}$$

It can be proved that the determinant value of above matrix gives Catalan Number.

$$|M_n| = 2\binom{2n-1}{n-1} - \frac{2n}{n+1}\binom{2n-1}{n-1},$$

which on simplification takes up the form

$$2\binom{2n-1}{n-1} = 2\left[\frac{(2n-1)!}{(n-1)!*n!}\right] = \frac{2n(2n-1)!}{n*(n-1)!*n!} = \frac{(2n)!}{n!*n!} = \binom{2n}{n}$$ ……….(6)

And the second part is,

$$\frac{2n}{n+1}\binom{2n-1}{n-1} = \frac{2n}{n+1}\left[\frac{(2n-1)!}{(n-1)!*n!}\right] = \frac{1}{n+1}\left[\frac{(2n)!}{n!*(n-1)!}\right] = \frac{(2n)!}{(n-1)!*(n+1)!} = \binom{2n}{n-1}$$ ………(7)

Combining both parts we have

$$|M_n| = \binom{2n}{n} - \binom{2n}{n-1} = C_n,\ n \in N$$

i.e. we have proved that,

$$\begin{vmatrix} \binom{2n-1}{n-1} & \binom{2n-1}{n-1} \\ \frac{2n}{n+1} & 2 \end{vmatrix} = C_n \,, n \in N \qquad \text{……….} (8)$$

2.1.2 Different Form of Catalan Numbers

Using Pascal's Triangle which shows the proceedings of different integral powers of binomial expansion is tabulated below and using that we have certain results on Catalan numbers.

(1) $2\binom{2n-1}{n} - \binom{2n}{n-1} = C_n,\ n \in N$

(2) $\binom{2n-1}{n} - \binom{2n-1}{n+1} = C_n,\ n \geq 2$

[These two results can also be established using laws of combination.]

2.2 Product of two successive Catalan Numbers & Pascal's Triangle

2.2.1 Pascal's Triangle

Pascal's triangle, named after French mathematician Blaise Pascal, is a triangular array of natural numbers. Each entry of a triangle is a binomial coefficient found.in the binomial expansion of $(a+b)^n$ where $n \in N.$ The Construction of Pascal's Triangle can be done; which is widely known.

Table 1: Binomial Coefficients

N																	
0									1		0						
1								1		1							
2							1		2		1						
3						1		3		3		1					
4					1		4		6		4		1				
5				1		5		10		10		5		1			
6			1		6		15		20		15		6		1		
7		1		7		21		35		35		21		7		1	
8	1		8		28		56		70		56		28		8		1

2.2.2 Product of Consecutive Catalan Numbers from Pascal's Triangle

We now discuss an important derivation obtained using coefficients of Pascal's triangle.
Consider the highlighted values in above Pascal's Triangles to form a Matrices of order 2.

Let $M_1 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$, $M_2 = \begin{bmatrix} 2 & 1 \\ 6 & 4 \end{bmatrix}, M_3 = \begin{bmatrix} 6 & 4 \\ 20 & 15 \end{bmatrix}, M_4 = \begin{bmatrix} 20 & 15 \\ 70 & 56 \end{bmatrix}, \dots$

Continuing this way, the general form of above matrices can be given as,

$$M_n = \begin{bmatrix} \binom{2n-2}{n-1} & \binom{2n-2}{n-2} \\ \binom{2n}{n} & \binom{2n}{n-1} \end{bmatrix}$$

On finding the determinant value of above Matrices, we observe that each determinant value represents product of two consecutive Catalan Numbers; which are the members of the Catalan sequence $C_i = 1,1,2,5,14,42, \dots$ Etc. for all $i \in N \cup \{0\}$

Det. M_1 = $|M_1| = \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} = 1 = C_1 * C_0 = 1 * 1$

$$|M_2| = \begin{vmatrix} 2 & 1 \\ 6 & 4 \end{vmatrix} = 2 = C_2 * C_1 = 2 * 1$$

$$|M_3| = \begin{vmatrix} 6 & 4 \\ 20 & 15 \end{vmatrix} = 10 = C_3 * C_2 = 5 * 2$$

$$|M_4| = \begin{vmatrix} 20 & 15 \\ 70 & 56 \end{vmatrix} = 70 = C_4 * C_3 = 14 * 5$$

The first term of each product representation of determinant value is Catalan Numbers C_n for $n = 1,2,3,\ldots$

For Generalization we can state that, For $n \in N - \{2\}$,

$|M_n| = C_n * C_{n-1}$ Where M_n is a Matrix of order 2 define as,

$$M_n = \begin{bmatrix} \binom{2n-2}{n-1} & \binom{2n-2}{n-2} \\ \binom{2n}{n} & \binom{2n}{n-1} \end{bmatrix} \qquad \ldots\ldots\ldots (9)$$

For illustration take $n = 3$

$$|M_3| = \begin{vmatrix} \binom{4}{2} & \binom{4}{1} \\ \binom{6}{3} & \binom{6}{2} \end{vmatrix} = \begin{vmatrix} 6 & 4 \\ 20 & 15 \end{vmatrix} = 10 = C_3 * C_2 = 5 * 2$$

2.3 Product Form, Bounds, and Ratio – a Limiting Case

At this stage we add two more points in the list. We have associated the general formula of Catalan number with all consecutive terms beginning with 2 prior to the general term

In the second part we have, in order to find the incremental rate in the broader case, taken the ratio and established that it tends to ‘4’ as the number of terms tend to infinity. This also helps us fix an upper bound to an nth term of Catalan sequence.

There is a recurrence relation connecting an n^{th}Catalan number C_n with its predecessor term C_{n-1} for all $n \geq 2 \ with \ C_1 = 1.$

$$C_n = \left(\frac{4n-6}{n}\right) C_{n-1}$$

[Above relation can be proved easily using Principal of Mathematical Induction.]

Using $C_n = \left(\frac{4n-6}{n}\right) C_{n-1}$ with $C_1 = 1$,

For $n = 2$ we get $C_2 = \frac{(4*2)-6}{2} = 1$

Extending this relationship for $n = 3, C_3 = \frac{(4*3)-6}{3} * \frac{(4*2)-6}{2}$ = 2

In the same way, for $n = 4, C_4 = \frac{(4*4)-6}{4} * \frac{(4*3)-6}{3} * \frac{(4*2)-6}{2}$ = 5

Continuing on the same line, we have $C_4 = \prod_{k=2}^{4} \left(\frac{(4*k)-6}{k}\right)$

Without loss of generality, this can be rewritten as,

$$C_n = \prod_{k=2}^{n} \left(4 - \frac{6}{k}\right) \qquad \ldots\ldots\ldots (10)$$

Therefore

$$C_n = 2^{n-1} \prod_{k=2}^{n} \left(2 - \frac{3}{k}\right)$$

Again

$$C_n = 2^{2n-2}\prod_{k=2}^{n}\left(1-\frac{3}{2k}\right) \qquad \text{………. (11)}$$

Note

Each factor $\left(1-\frac{3}{2k}\right)$ [for every value of $k \geq 2$] is less than unity

$$\Rightarrow \prod_{k=2}^{n}\left(1-\frac{3}{2k}\right) < 1$$

$$\Rightarrow C_n = 2^{2n-2}\prod_{k=2}^{n}\left(1-\frac{3}{2k}\right) < 2^{2n-2}$$

Which is an upper bound of C_n for all value of $n \in N$
From (11) we have

$$C_n = 2^{2n-2}\prod_{k=2}^{n}\left(\frac{2k-3}{2k}\right)$$

$$= 2^{2n-2}\left[\frac{\prod_{k=2}^{n} 2k-3}{\prod_{k=2}^{n} 2k}\right]$$

$$\therefore C_n = \frac{2^{n-1}}{n!}\prod_{k=2}^{n}(2k-3) \qquad \text{………. (12)}$$

Replacing (n+1)for n in (12) we get

$$C_{n+1} = \frac{2^{n}}{(n+1)!}\prod_{k=2}^{n+1}(2k-3) \qquad \text{………. (13)}$$

Searching for the limiting value of the ratio as discussed above, we proceed as follows.
Using Equations (12) and (13)

$$\lim_{n\to\infty}\frac{C_{n+1}}{C_n} = \lim_{n\to\infty}\frac{\frac{2^{n}}{(n+1)!}\prod_{k=2}^{n+1}(2k-3)}{\frac{2^{n-1}}{n!}\prod_{k=2}^{n}(2k-3)}$$

$$= \lim_{n\to\infty}\frac{2}{n+1}(2n-1)$$

$$= \lim_{n\to\infty}\frac{2\left(2-\frac{1}{n}\right)}{\left(1+\frac{1}{n}\right)}$$

Therefore, we have

$$\lim_{n\to\infty}\frac{C_{n+1}}{C_n} = 4 \qquad \text{………. (14)}$$

To show graphical presentation we draw the graph of corresponding ratio of term to its previous term. The graph indicates the increasing trend of ratio and as the terms increase. It, as established above, tends to the numeric value ‘4’.
The ratio of a term to its successive terms as the number of terms advances can be graphed and can be shown as below.

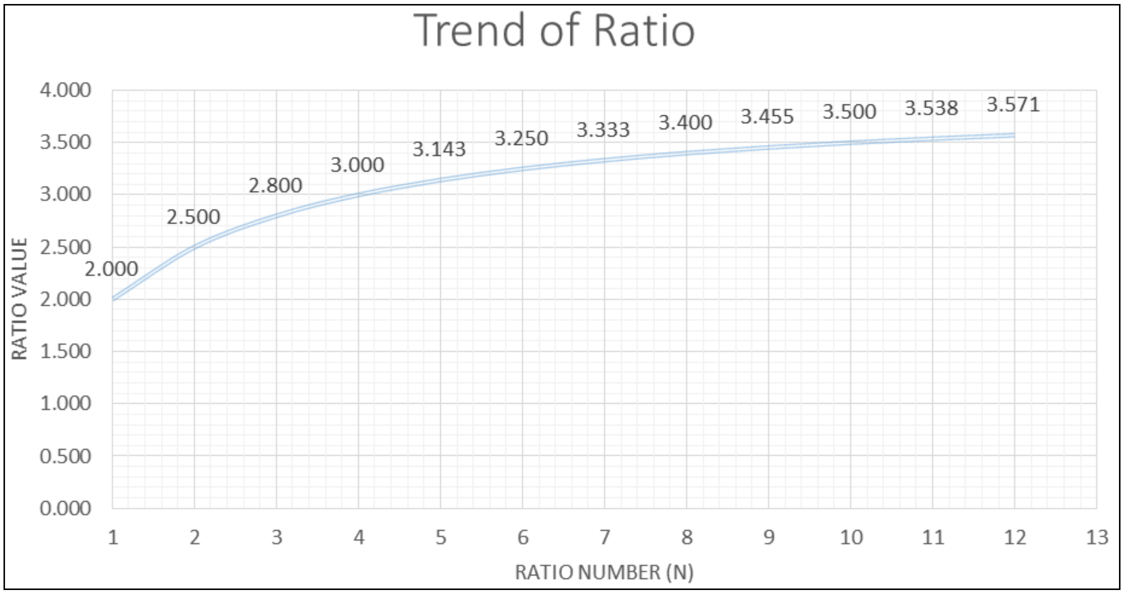

Fig 2

3. Relation between Narayana numbers and Catalan numbers

3.1 Introduction to Narayana Numbers

In this section we establish a relation between Narayana numbers with Catalan numbers. We begin with restructuring the defining formula of Narayan numbers and then establish its mathematical relation with Catalan numbers.

Narayana Numbers are derived as a result of the formula,

$N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$ Where, n, and k are natural numbers with $n \geq k$

$$N(n,k) = \frac{1}{n}\left[\left(\frac{n!}{k!\,(n-k)!}\right)\left(\frac{n!}{(k-1)!\,(n-k+1)!}\right)\right]$$

$$= \frac{1}{n}\left[\left(\frac{(n)(n-1)(n-2)\ldots(n-k+1)}{k!}\right)\left(\frac{(n)(n-1)(n-2)\ldots(n-k+2)}{(k-1)!}\right)\right]$$

$$= \frac{k}{n}\left(\frac{\prod_{p=0}^{k-2}(n-p)}{k!}\right)^2 (n-k+1)$$

$$\therefore N(n,k) = \frac{k(n-k+1)}{n}\left(\frac{\prod_{p=0}^{k-2}(n-p)}{k!}\right)^2 \qquad \text{......... (15)}$$

We now work on deriving connection of Narayana numbers with Catalan numbers.

Let $N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$ .... (16)

3.2 Even and Odd Forms

Put $n = 2k$ in equation (16)

$$N(2k,k) = \frac{1}{n}\binom{2k}{k}\binom{2k}{k-1}$$

$$= \frac{1}{2k}\left[\left(\frac{(2k)!}{k!\,k!}\right)\left(\frac{(2k)!}{(k-1)!\,(k+1)!}\right)\right]$$

$$= \frac{1}{2k}\left[\left(\frac{(2k)!}{k!\,k!}\right)\left(\frac{2k(2k-1)!}{(k-1)!\,(k+1)!}\right)\right]$$

$$= \left[\left(\frac{(2k)!}{k!\,k!}\right)\left(\frac{(2k-1)!}{(k-1)!\,(k+1)k!}\right)\right]$$

$$= \frac{1}{k+1}\left(\frac{(2k)!}{k!\,k!}\right)\left(\frac{(2k-1)!}{(k-1)!\,k!}\right)$$

$$\therefore N(2k,k) = C_k\binom{2k-1}{k} \quad \left(C_k = \frac{1}{k+1}\binom{2k}{k} = \frac{1}{k+1}\left(\frac{2k!}{k!k!}\right)\right)$$

We have proved that

$N(2k,k) = C_k\binom{2k-1}{k}$ Where C_k is the k^{th} Catalan number ……… (17)

Again on putting $n = 2k-1$ in equation (16), we have

$$N(2k-1,k) = \frac{1}{2k-1}\binom{2k-1}{k}\binom{2k-1}{k-1}$$
$$= \frac{1}{2k-1}\left[\left(\frac{(2k-1)!}{k!\,(k-1)!}\right)\left(\frac{(2k-1)!}{(k-1)!\,k!}\right)\right]$$
$$= \frac{1}{2k-1}\left[\left(\frac{(2k-1)(2k-2)!}{k!\,(k-1)!}\right)\left(\frac{(2k-1)!}{(k-1)!\,k!}\right)\right]$$
$$= \left(\frac{(2k-2)!}{k!\,(k-1)!}\right)\left(\frac{(2k-1)!}{(k-1)!\,k!}\right) = \left(\frac{(2k-2)!}{(k-1)!\,(k-1)!}\right)\left(\frac{(2k-1)!}{k!\,k!}\right)$$
$$= \left(\frac{1}{k}\left(\frac{(2k-2)!}{(k-1)!\,(k-1)!}\right)\right)\left(\frac{(2k-1)!}{(k-1)!\,k!}\right)$$
$$\therefore N(2k-1,k) = C_{k-1}\binom{2k-1}{k} \qquad \left(C_{k-1} = \frac{1}{k}\binom{2k-2}{k-1} = \frac{1}{k}\left(\frac{(2k-2)!}{(k-1)!\,(k-1)!}\right)\right)$$

We have proved that,

$N(2k-1,k) = C_{k-1}\binom{2k-1}{k}$ Where C_{k-1} is the $(k-1)^{th}$ Catalan Number ………. (18)

We have proved two results,

$N(2k,k) = C_k\binom{2k-1}{k}$ Where C_k is the k^{th} Catalan number

$N(2k-1,k) = C_{k-1}\binom{2k-1}{k}$ Where C_{k-1} is the $(k-1)^{th}$ Catalan number

We give an illustration to this point.
For $k = 2$, using above two results we get

$N(4,2) = C_2\binom{3}{2} = C_2 * 3 = 2 * 3 = 6$ and $N(3,2) = C_1\binom{3}{2} = C_1 * 3 = 1 * 3 = 3$

4. Conclusion

It is known that there are many approaches and feasible interpretations for Catalan numbers and our efforts are in the same direction. Efforts in establishing connection between Catalan numbers and Narayana numbers have signaled meaningful interpretations.

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