



Part B: Characteristics and special role of matrices of Class 4B

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Abstract

We have classified square matrices based on the 'Libra value' * property in six different classes. In this note we unfold what we have found and establish about dominating properties of matrices of class 4B** and also its ring structure. It has been shown that some sets of standard form of class 4B matrices satisfy commutative property under matrix multiplication. In addition to this, we describe different transformations that convert matrices of class 4B to their corresponding matrices of class 4A and vice – versa.

Libra value: Libra value of a given square matrix is that constant value for the given matrix obtained by adding the all the entries of each row or column or diagonal (both leading or non-leading) as the case may be depending on the classification.

Notations: (1) CJ4A ($n \times n$, $L(A) = _$) [Square Matrix of class 4A with order $n \times n$ and $L(A) =$ libra value. $L(A) =$ libra value]

(2) CJ4B ($n \times n$, $L(A) = _$) [Square matrix of class 4B with order $n \times n$] and $L(B) =$ libra Value]

(3) CJS4B [A symmetric matrix of CJ4B]

(4) $\bar{0}$ is a Null matrix

(5) I is an identity matrix

Keywords: class – 4B, class 4A, libra value, eigen values

Theorem 2

To prove that the infinite set of the matrices of the above format denoted as CJ4B* is a field under the two binary operations '+' and '.' (Regular matrix operations). i.e. (CJ4, '+', '.') is a field.

Proof: In the section (3) we have already established that (CJ4B*, '+') is a commutative group under binary operation '+'. Now we extend the notion gradual and move in the targeted direction.

(1) Closure Property: i.e. '.' is a binary operation:

We have the set CJ4. let A, and B \in CJ4

For some $p_1, p_2, k_1,$ and k_2 we have

$$A = \begin{bmatrix} p_1 + k_1 & p_1 - k_1 & p_1 + k_1 \\ p_2 + k_2 & p_2 & p_2 \\ p_1 - k_1 & p_1 + 2k_1 & p_1 \end{bmatrix} \in CJ4(3 \times 3, L(A) = 3p_1 + k_1)$$

and

$$B = \begin{bmatrix} p_2 + k_2 & p_2 - k_2 & p_2 + k_2 \\ p_2 + k_2 & p_2 & p_2 \\ p_2 - k_2 & p_2 + 2k_2 & p_2 \end{bmatrix} \in CJ4(3 \times 3, L(B) = 3p_2 + k_2)$$

We have $p_1, p_2, k_1,$ and k_2 are real values

$$\text{So that } L(A) = 3p_1 + k_1 \neq 0 \text{ and } L(B) = 3p_2 + k_2 \neq 0$$

Then the product of A by the matrix B, denoted as AB is also a matrix of the same class. [Generally '.' is omitted as the togetherness of the symbols means the product.]

i.e. $AB \in CJ4$

[An important point to note is that two matrices of class 4B result into a matrix of class CJ4A but still it remains in the set CJ4.]

$$L(AB) = L(A) \cdot L(B)$$

(2) ‘.’ is associative

It can be shown that for the matrices A, B, and C of the class CJ4B* are such that they obey A (BC) = (AB)C [for A, B and C ∈ CJ4B. A = (a_{ij}), B = (b_{ij}), and C = (c_{ij}) are of the same order.

Let L(A) = p1, L(B) = p2, and L(C) = p3. all real entries.

Since a_{ij}, b_{ij}, and c_{ij} are real entries and associative law for multiplication holds on the set of real values in R]

We have A (B C) = (A B) C ... (22)

(3) ‘.’ distributive over ‘+’

For the matrices A, B, and C under the same format (members of class 4B) satisfy doubly distributive laws.

I.e. A (B + C) = AB + AC ... (23)

And (B + C) A = BA + CA

[Note: Properties (1), (2), and (3) with ‘.’ as binary operation along with the previously proved result that CJ4B* is an Abelian group under ‘+’ imply that (CJ4B*, ‘+’, ‘.’) is Ring

... (24)

(4) Commutative Ring

Also it can be established that (CJ4B*, ‘+’, ‘.’) is an Abelian/ Commutative Ring under ‘.’

i.e. for $A \in CJ4(3 \times 3, L(A) = 3p_1 + k_1)$
 $B \in CJ4(3 \times 3, L(B) = 3p_2 + k_2)$

With p₁, p₂, k₁, and k₂ are such that **AB = BA**

$AB \in CJ4A(3 \times 3, L(AB) = (3p_1 + k_1)(3p_2 + k_2))$... (25)

[For the proof of the above result, please refer to the annexure1]

Now this will force call (CJ4B*, ‘+’, ‘.’) a commutative Ring.

At this critical stage, we give a numeric example.

Let $A \in CJ4B(3 \times 3, L(A) = 3p_1 + k_1)$ and $B \in CJ4B(3 \times 3, L(B) = 3p_2 + k_2)$

Of the from $A = \begin{bmatrix} p+k & p-k & p+k \\ p+k & p & p \\ p-k & p+2k & p \end{bmatrix}$

Let $A = \begin{bmatrix} 5 & -1 & 5 \\ 5 & 2 & 2 \\ -1 & 8 & 2 \end{bmatrix} \in CJ4B(3 \times 3, L(A) = 9)$ For p = 2 and k = 3

and $B = \begin{bmatrix} 9 & -1 & 9 \\ 9 & 4 & 4 \\ -1 & 14 & 4 \end{bmatrix} \in CJ4B(3 \times 3, L(B) = 17)$ For p = 4, k = 5

Then $AB = BA = \begin{bmatrix} 31 & 61 & 61 \\ 61 & 31 & 61 \\ 61 & 61 & 31 \end{bmatrix} \in CJ4B(3 \times 3, L(AB) = L(BA) = 153)$

Also note that $L(AB) = L(BA) = L(A)L(B) = 153 = L(A) = 9. L(B) = 17$

3.3 Ring with Unity

We have defined the set CJ4B* = CJ4B ∪ I Where I is an identify matrix.

This means that (CJ4B*, ‘+’, ‘.’) is a commutative Ring with unity.

4. Eigen Values:

The role of eigen value and eigen vectors in the matrix system and as its application to real life situation is very important. These are also called ‘latent values’ associated with a given matrix. For a given matrix A if we find a non-zero vector **X** and a real value λ such that $A\mathbf{X} = \lambda\mathbf{X}$ then λ is called the eigen value and the vector **X** is called the eigen vector.

In our system of matrices of class4B, we have a very important property that one of the eigen value is the libra value of the given matrix. We prove this statement.

Theorem 3

Statement: One of the Eigen values of matrices of class 4A is its libra value.

Proof: Let us consider a matrix of order nxn of the class 4A.

$$\text{Let } A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & p1 \\ a_{21} & a_{22} & a_{23} & \dots & p2 \\ a_{31} & a_{32} & a_{33} & \dots & p3 \\ \dots & \dots & \dots & \dots & \dots \\ a_{(n-1)1} & a_{(n-1)2} & a_{(n-1)3} & \dots & p_{(n-1)} \\ s1 & s2 & s3 & \dots & \alpha \end{pmatrix} \in \text{CJ4A } (nxn, L(A) = p) \text{ where } p \in R \tag{28}$$

Also implied by properties of class4A, we have the following constraints.

$$\begin{aligned} s1 &= p - (a_{11} + a_{21} + \dots + a_{(n-1)1}), p1 = p - (a_{11} + a_{12} + \dots + a_{1(n-1)}) \\ s2 &= p - (a_{12} + a_{22} + \dots + a_{(n-1)2}), p2 = p - (a_{21} + a_{22} + \dots + a_{2(n-1)}) \\ s3 &= p - (a_{13} + a_{23} + \dots + a_{(n-1)3}), p3 = p - (a_{31} + a_{32} + \dots + a_{3(n-1)}) \\ \text{And } \alpha &= p - (s1 + s2 + s3 + \dots + s(n-1)) = p - (p1 + p2 + p3 + \dots + p(n-1)) \end{aligned}$$

Also the sum of non-leading diagonal entries = $p = L(A)$

Now for any non-zero vector $\mathbf{X}_{n \times 1}$ and a real λ , we write $A\mathbf{X} = \lambda\mathbf{X}$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & q1 \\ a_{21} & a_{22} & a_{23} & \dots & q2 \\ a_{31} & a_{32} & a_{33} & \dots & q3 \\ \dots & \dots & \dots & \dots & \dots \\ a_{(n-1)1} & a_{(n-1)2} & a_{(n-1)3} & \dots & q_{(n-1)} \\ t1 & t2 & t3 & \dots & \alpha \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \dots \\ x_{n-1} \\ x_n \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \dots \\ x_n \end{pmatrix}$$

We write these equations below.

$$\begin{aligned} a_{11} x_1 + a_{12} x_2 + a_{13} x_3 + \dots + q1 x_n &= \lambda x_1 \\ a_{21} x_1 + a_{22} x_2 + a_{23} x_3 + \dots + q2 x_n &= \lambda x_2 \\ a_{31} x_1 + a_{32} x_2 + a_{33} x_3 + \dots + q3 x_n &= \lambda x_3 \\ \dots & \dots \\ t1 x_1 + t2 x_2 + t3 x_3 + \dots + \alpha x_n &= \lambda x_n \end{aligned}$$

Where t1, t2, t3... q1, q2, q3... and α are defined as above.

Adding these entries column wise, and using relations for s1, s2, s3... p1, p2, p3,....., and α we get

$$px_1 + px_2 + px_3 + \dots + px_n = \lambda (x_1 + x_2 + x_3 + \dots + x_n)$$

As p, and λ are real values, we have $p\mathbf{X} = \lambda\mathbf{X}$ and X being non-zero vector, the result $p = \text{libra value} = \lambda$ follows.

We conclude that a libra value, L(A), of the matrix is one of the eigen value and hence the proof.

As a consequence of the above theorem, we have In our case of the matrix of class 4B, given by the result (16)

$$A = \begin{bmatrix} p+k & p-k & p+k \\ p+k & p & p \\ p-k & p+2k & p \end{bmatrix}_{3 \times 3} \in \text{CJ4B}^*$$

With $L(A) = p = 3p + k$,

Eigen values are (1) $3p+k$ (= $L(A)$ = libra value) (2) $\sqrt{2} k i$ and (3) $-\sqrt{2} k i$ (29)

Eigen vector corresponding to $L(A) = 3p+k$ is $(1,1,1)' = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

[Note: We note two important observations.

1. Sum of eigen values = trace of the matrix $A = \text{Tr.}(A) = \text{sum of all entries of leading diagonal} = 3p + k = L(A)$ of matrix A of class 4B.
2. In the case of a matrix of order 3×3 (or of any order $n \times n$) of the class 4B, once an eigen value, Depending on libra value is known then it is simpler to find remaining eigen values.
- 3 **Fact:** If for a given non-singular matrix of class 4B in the above form two eigen values are imaginary and their corresponding eigen vectors too. Eigen values corresponding to the symmetric form matrix of the class 4B are all real. We elaborate the fact.

A symmetric matrix given by the form (19)

$$A = \begin{bmatrix} p-k & p-m & p+m+2k \\ p-m & p+m+2k & p-k \\ p+m+2k & p-k & p-m \end{bmatrix} \text{ with } L(A) = 3p+k$$

has all real eigen values.

These values are (1) $3p+k$ (2) $\sqrt{7k^2 + 6km + 3m^2}$ (3) $-\sqrt{7k^2 + 6km + 3m^2}$

Note that $L(A) = 3p+k =$ one of the eigen values.

The eigen vector corresponding to $3p+k$ is $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

For the symmetric matrix $A = \begin{pmatrix} 2 & 4 & 3 \\ 4 & 3 & 2 \\ 3 & 2 & 4 \end{pmatrix}$ for which, in the form (19), $p = 11$, $k = 3/4$, and $m = -5/4$

The eigen values are (1) $3p+k = 9$ (2) $\sqrt{3}$ and (3) $-\sqrt{3}$

5. Graphical Presentation:

In this part we make graphical presentation of column vector assuming each one independently stands to represent a quadratic parabola. On plotting each one of the column vector corresponding to a 3×3 matrix of class 4B on two dimensional rectangular frame, we can clearly understand the notion of libra value associated with a matrix and many more features too. We consider a numerical illustration.

Let us consider the above mentioned 3×3 matrix of class 4BS.

Let $A = \begin{pmatrix} 2 & 4 & 3 \\ 4 & 3 & 2 \\ 3 & 2 & 4 \end{pmatrix}$ for which, in the form (19), $p = 11/4$, $k = 3/4$, and $m = -5/4$ and $L(A) = 9$

Let each column vector represent a quadratic curve designed in the following way.

$Y1 = 2 + 4x + 3x^2$, $Y2 = 4 + 3x + 2x^2$, and $Y3 = 3 + 2x + 4x^2$. We plot Y values for different Values of x.

We plot these three graphs on the same plotting scaled axis.

Table 1

X	y1	y2	y3
-2	4	16	10
-1	2	4	4
0	4	2	4
1	10	10	10
2	20	28	22

The graph is shown below.

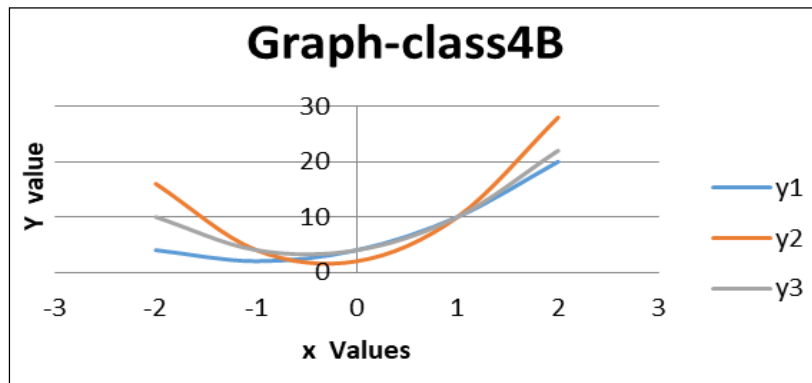


Fig 1

It is clear from the graph that $(x=1, y=9)$ is the common point of intersection where all the graphs intersect. The y ordinate of the common point of intersection is the libra value= $L(A)=9$

5.1 A Finite Sequence of $L(A) = 3p+1$ converging in to $L(A) = 4, 2, 1$:

[Note that in all cases $3p + 1$ is either an even or an odd integer.]

It is a well-known factual game and still it has many points unsolvable. We draw its salient features.

1. One begins with any positive integer
2. If it is an odd one multiply the integer by 3 and then add one.
3. If it is an even one divide by 2.
4. Apply any one of the above rule at each stage until we get a convergent sequence of integers 8, 4, 2, 1.

We follow the same pattern on a matrix of class 4B. and finally **any** matrix **A** of class 4B of the type

$$A = \begin{pmatrix} p+k & p-k & p+k \\ p+k & p & p \\ p-k & p+2k & p \end{pmatrix} \in CJ4B \text{ With } L(A) = 3p+k,$$

With $k=1$ and any integer p will be converted into a matrix in the form

$$F = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ -1 & 2 & 0 \end{pmatrix} \tag{30}$$

[This is elaborated in annexure-2]

6. Conclusion

Detailed study of each class through 1 up to 4B has established that the more one gets involved in the study of finer properties of the classes of square matrices, he meets with invincible properties and gets more avenues open to further studies.

7. Vision

This study has opened one more additional aspect that involves combination of all the properties inherent in all the classes those have been previously studied. The next class that is the class 5 – to some extent one can call magic squares—is not limited to mathematics for the purpose of fun. There are many algebraic properties and subtle techniques of extension from matrices of order 3×3 to $n \times n$. These techniques vary in the cases for even and odd order. This is our next target in the immediate paper that follows.

8. References

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Annexure-1

In this part we show that for the class 4B of matrices A, and B in the format as shown below commutative property for matrix multiplication is observed.

$$\text{Let } \begin{pmatrix} p+k & p-k & p+k \\ p+k & p & p \\ p-k & p+2k & p \end{pmatrix} \in CJ4B \text{ With } L(A) = 3p+k,$$

We consider p1, p2, k1, and k2 (all real values and prove that AB = BA.

$$\text{Let } A = \begin{pmatrix} p1+k1 & p1-k1 & p1+k1 \\ p1+k1 & p1 & p1 \\ p1-k1 & p1+2k1 & p1 \end{pmatrix} \in CJ4B \text{ With } L(A) = 3p1+k1,$$

$$\text{And let } B = \begin{pmatrix} p2+k2 & p2-k2 & p2+k2 \\ p2+k2 & p2 & p2 \\ p2-k2 & p2+2k2 & p2 \end{pmatrix} \in CJ4B \text{ With } L(A) = 3p2+k2$$

Both the matrices are conformable for matrix multiplication. We find AB and BA by actual working on laid down rules. The results are as follows.

$$AB = \begin{pmatrix} -k1k2+k1p2+k2p1+3p1p2 & k1k2+k1p2+k2p1+3p1p2 & k1k2+k1p2+k2p1 \\ k1k2+k1p2+k2p1+3p1p2 & -k1k2+k1p2+k2p1+3p1p2 & k1k2+k1p2+k2p1 \\ k1k2+k1p2+k2p1+3p1p2 & k1k2+k1p2+k2p1+3p1p2 & -k1k2+k1p2+k2p1 \end{pmatrix}$$

We get the same result for BA. Hence under the *monotype of matrices of higher order of class 4B* too, we conclude that, in general **AB = BA i.e. in general Commutative property holds true in matrix multiplication.**

Annexure 2

As discussed for the matrix

$$A = \begin{pmatrix} p+k & p-k & p+k \\ p+k & p & p \\ p-k & p+2k & p \end{pmatrix} \in CJ4B \text{ With } L(A) = 3p+k,$$

with libra value L(A) = 3p+k, where p and k, initially, are integers.

(1) We begin by taking p = 3 and k = 1,

$$\text{The first matrix } A = A = \begin{pmatrix} 4 & 2 & 4 \\ 2 & 3 & 3 \\ 2 & 5 & 3 \end{pmatrix} L(A) = 3(p=3) + 1 = 10$$

L(A) = 10 which is even; so we look for a matrix with libra value = 5

Let 3p1 + 1 = 5. This implies that p1 = 4/3. And the fix value k = 1 gives the new matrix in the same Format as follows.

$$A1 = \begin{pmatrix} 7/3 & 1/3 & 7/3 \\ 7/3 & 4/3 & 4/3 \\ 1/3 & 10/3 & 4/3 \end{pmatrix} L(A1) = 3(p1 = 4/3) + 1 = 5$$

Now, L(A1) = 5 which is odd and so by the rule we multiply it by 3 and add 1 to get the new libra value = 3(p2) + 1 = 16 this implies that p2 = 5.

The new matrix for this, say A2, is as follows.

$$A2 = \begin{pmatrix} 6 & 4 & 6 \\ 6 & 5 & 5 \\ 4 & 7 & 5 \end{pmatrix} L(A2) = 3(p1 = 5) + 1 = 16$$

Now, L(A2) = 16 which is even and so by the rule we divide it by 2 to get the new libra value = 3(p3) + 1 = 8 this implies that p3 = 7/3.

The new matrix for this, say A3, is as follows.

$$A_3 = \begin{pmatrix} 10/3 & 4/3 & 10/3 \\ 10/3 & 7/3 & 7/3 \\ 4/3 & 13/3 & 7/3 \end{pmatrix} L(A_3) = 3(p_4) + 1 = 8$$

Now, $L(A_3) = 8$ which is even and so by the rule we divide it by 2 to get the new libra value $= 3(p_4) + 1 = 4$ this implies that $p_4 = 1$.

The new matrix for this, say A_4 , is as follows.

In the same way we continue and finally we end with a sequence 8, 4, 2, and 1.

The last matrix with libra value = 1 which implies that $p = 0$.

This has a final matrix P which is necessarily of the following form which is already given by form (29)

$$F = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ -1 & 2 & 0 \end{pmatrix} L(P) = 3(p=0) + 1 = 1$$

This logic of $3p + 1$ works for all matrices of the standard form and forcing them to end up in the above matrix F .