International Journal of Advanced Science and Research
ISSN: 2455-4227; Impact Factor: RJIF 5.12
Received: 26-01-2019; Accepted: 01-03-2019
www.allsciencejournal.com
Volume 4; Issue 3; May 2019; Page No. 01-05

On the Pellian like Equation $3x^2 – 7y^2 = 20$

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Abstract
The binary quadratic equation represented by the Pellian like equation $3x^2 – 7y^2 = 20$ is analyzed for its distinct integer solutions. A few interesting relations among the solutions are given. Employing the solutions of the above hyperbola, we have obtained solutions of other choices of hyperbolas and parabolas.

Keywords: binary quadratic, hyperbola, parabola, Pell equation, integer solutions

Introduction

The binary quadratic Diophantine equation of the form $ax^2 - by^2 = N, \rightleftarrows (a, b, N \neq 0)$ are rich in variety and have been analyzed by many mathematicians for their respective integer solutions for particular values of a, b and N. In this context, one may refer [1-15]

This communication concerns with the problem of obtaining non – zero distinct integer solutions to the binary quadratic equation given by $3x^2 - 7y^2 = 20$ representing hyperbola. A few interesting relations among its solutions are presented. Knowing an integral solution of the given hyperbola, integer solutions for other choices of hyperbolas and parabolas are presented.

Method of analysis

The Diophantine Equation representing the binary quadratic equation to be solved for its non - zero distinct integral solution is

$$3x^2 - 7y^2 = 20 \tag{1}$$

Consider the linear transformation

$$x = X + 7T \ , \ y = X + 3T \tag{2}$$

From (1) and (2), we have

$$X^2 = 21T^2 - 5 \tag{3}$$

whose smallest positive integer solution is

$$T_0 = 1 X_0 = 4$$

To obtain the other solutions of (3), consider the pellian equation is

$$X^2 = 21T^2 + 1 \tag{4}$$

whose smallest positive integer solution is

$$\tilde{T}_0 = 12 \tilde{X}_0 = 55$$

The general solution of (4) is give by

$$\tilde{T}_n = \frac{1}{2\sqrt{21}} g_n$$

$$\tilde{X}_n = \frac{1}{2} f_n$$

Where

$$f_n = \left(55+12\sqrt{21}\right)^{n+1} + \left(55-12\sqrt{21}\right)^{n+1}, g_n = \left(55+12\sqrt{21}\right)^{n+1} - \left(55-12\sqrt{21}\right)^{n+1}, n = -1,0,1\ldots$$

Applying Brahmagupta lemma between (T_0, X_0) and $(\tilde{T}_n, \tilde{X}_n)$, the other integer solutions of (1) are given by

$$T_{n+1} = \frac{1}{2}f_n + \frac{2}{\sqrt{21}}g_n \qquad (5)$$

$$X_{n+1} = 2f_n + \frac{\sqrt{21}}{2}g_n \qquad (6)$$

From (2), (4), (5) and (6) the values of x and y satisfying given by

$$x_{n+1} = \frac{11}{2}f_n + \frac{49}{2\sqrt{21}}g_n$$

$$y_{n+1} = \frac{7}{2}f_n + \frac{33}{2\sqrt{21}}g_n$$

The recurrence relations satisfied by the values of x and y are given by

$$x_{n+3} - 110x_{n+2} + x_{n+1} = 0,$$

$$y_{n+3} - 110y_{n+2} + y_{n+1} = 0, n = -1,0,1\ldots$$

Some numerical examples of x and y satisfying (1) are given in the Table:1 below:

Table 1: Numerical Examples

n	x_{n+1}	y_{n+1}
-1	11	7
0	1193	781
1	131219	85903
2	14432897	9448549
3	1587487451	1039254487
4	174609186713	114308545021

A few interesting relations among the solutions are given below:

1. x_{n+1} **and** y_{n+1} **are always odd**
2. **Relations among the solutions**
   - $55x_{n+1} - x_{n+2} + 84y_{n+1} = 0$
   - $x_{n+1} - 55x_{n+2} + 84y_{n+2} = 0$
   - $55x_{n+1} - 6049x_{n+2} + 84y_{n+3} = 0$
   - $6049x_{n+1} - x_{n+3} + 9240y_{n+1} = 0$
   - $6049x_{n+2} - 55x_{n+3} + 84y_{n+1} = 0$
   - $x_{n+1} - x_{n+3} + 168y_{n+2} = 0$
   - $55x_{n+2} - x_{n+3} + 84y_{n+2} = 0$
   - $x_{n+1} - 6049x_{n+3} + 9240y_{n+3} = 0$
   - $x_{n+2} - 55x_{n+3} + 84y_{n+3} = 0$
   - $55y_{n+1} - y_{n+2} + 36x_{n+1} = 0$
   - $y_{n+1} - 55y_{n+2} + 36x_{n+2} = 0$
   - $55y_{n+1} - 6049y_{n+2} + 36x_{n+3} = 0$
   - $3960x_{n+1} + 6049y_{n+1} - y_{n+3} = 0$
   - $y_{n+1} - y_{n+3} + 72x_{n+2} = 0$

- $y_{n+1} - 6049y_{n+3} + 3960x_{n+3} = 0$
- $36x_{n+1} - 55y_{n+3} + 6049y_{n+2} = 0$
- $36x_{n+2} - y_{n+3} + 55y_{n+2} = 0$
- $36x_{n+3} - 55y_{n+3} + y_{n+2} = 0$

3. Each of the following expressions represents a Nasty Integer

- $\frac{1}{140}(5467x_{2n+2} - 49x_{2n+3} + 1680)$
- $\frac{1}{15400}(601321x_{2n+2} - 49x_{2n+4} + 184800)$
- $\frac{3}{5}(33x_{2n+2} - 49y_{2n+2} + 20)$
- $\frac{3}{275}(3579x_{2n+2} - 49y_{2n+3} + 1100)$
- $\frac{3}{30245}(393657x_{2n+2} - 49y_{2n+4} + 120980)$
- $\frac{1}{140}(601321x_{2n+3} - 5467x_{2n+4} + 1680)$
- $\frac{3}{275}(33x_{2n+3} - 5467y_{2n+2} + 1100)$
- $\frac{3}{5}(3579x_{2n+3} - 5467y_{2n+3} + 20)$
- $\frac{3}{275}(393657x_{2n+3} - 5467y_{2n+4} + 1100)$
- $\frac{3}{30245}(33x_{2n+4} - 601321y_{2n+2} + 120980)$
- $\frac{3}{275}(3579x_{2n+4} - 601321y_{2n+3} + 1100)$
- $\frac{3}{5}(393657x_{2n+4} - 601321y_{2n+4} + 20)$
- $\frac{1}{20}(11y_{2n+3} - 1193y_{2n+2} + 240)$
- $\frac{1}{2200}(11y_{2n+4} - 131219y_{2n+2} + 26400)$
- $\frac{1}{20}(1193y_{2n+4} - 131219y_{2n+3} + 240)$

4. Each of the following expressions represents a Cubical Integer

- $\frac{1}{92400}[601321x_{3n+3} - 49x_{3n+5} + 1803963x_{n+1} - 147x_{n+3}]$
- $\frac{1}{840}[5467x_{3n+3} - 49x_{3n+4} + 16401x_{n+1} - 141x_{n+2}]$
- $\frac{1}{10}[99x_{n+1} - 147y_{n+1} + 33x_{3n+3} - 49y_{3n+3}]$
- $\frac{1}{550}[10737x_{n+1} - 147y_{n+2} + 3579x_{3n+3} - 49y_{3n+4}]$
- $\frac{1}{60490}[1180971x_{n+1} - 147y_{n+3} + 393657x_{3n+3} - 49y_{3n+5}]$
- $\frac{1}{840}[1803963x_{n+2} - 16401x_{n+3} + 601321x_{3n+4} - 5467x_{3n+5}]$
- $\frac{1}{550}[99x_{n+2} - 16401y_{n+1} + 33x_{3n+4} - 5467y_{3n+3}]$
- $\frac{1}{10}[10737x_{n+2} - 16401y_{n+2} + 3579x_{3n+4} - 5467y_{3n+4}]$
- $\frac{1}{550}[1180971x_{n+2} - 16401y_{n+3} + 393657x_{3n+4} - 5467y_{3n+5}]$
- $\frac{1}{60490}[99x_{n+3} - 1803963y_{n+1} + 33x_{3n+5} - 601321y_{3n+3}]$
- $\frac{1}{550}[10737x_{n+3} - 1803963y_{n+2} + 3579x_{3n+5} - 601321y_{3n+4}]$
- $\frac{1}{10}[1180971x_{n+3} - 1803963y_{n+3} + 393657x_{3n+5} - 601321y_{3n+5}]$

- $\frac{1}{360}[99y_{n+2} - 10737y_{n+1} + 33y_{3n+4} - 3579y_{3n+3}]$
- $\frac{1}{39600}[99y_{n+3} - 1180971y_{n+1} + 33y_{3n+5} - 393657y_{3n+3}]$
- $\frac{1}{360}[10737y_{n+3} - 1180971y_{n+2} + 3579y_{3n+5} - 393657y_{3n+4}]$

5. Each of the following expressions represents a Bi-quadratic integer

- $\frac{1}{92400}[601321x_{4n+4} - 49x_{4n+6} + 2405284x_{2n+2} - 196x_{2n+4} + 554400]$
- $\frac{1}{840}[5467x_{4n+4} - 49x_{4n+5} + 21868x_{2n+2} - 196x_{2n+3} + 5040]$
- $\frac{1}{10}[33x_{4n+4} - 49y_{4n+4} + 132x_{2n+2} - 196y_{2n+2} + 60]$
- $\frac{1}{550}[3579x_{4n+4} - 49y_{4n+5} + 14316x_{2n+2} - 196y_{2n+3} + 3300]$
- $\frac{1}{60490}[393657x_{4n+4} - 49y_{4n+6} + 1574628x_{2n+2} - 196y_{2n+4} + 362940]$
- $\frac{1}{840}[601321x_{4n+5} - 5467x_{4n+6} + 2405284x_{2n+3} - 21868x_{2n+4} + 5040]$
- $\frac{1}{550}[33x_{4n+5} - 5467y_{4n+4} + 132x_{2n+3} - 21868y_{2n+2} + 3300]$
- $\frac{1}{10}[3579x_{4n+5} - 5467y_{4n+5} + 14316x_{2n+3} - 21868y_{2n+3} + 60]$
- $\frac{1}{550}[393657x_{4n+5} - 5467y_{4n+6} + 1574628x_{2n+3} - 21868y_{2n+4} + 3300]$
- $\frac{1}{60490}[33x_{4n+6} - 601321y_{4n+4} + 132x_{2n+4} - 2405284y_{2n+2} + 362940]$
- $\frac{1}{550}[3579x_{4n+6} - 601321y_{4n+5} + 14316x_{2n+4} - 2405284y_{2n+3} + 3300]$
- $\frac{1}{10}[393657x_{4n+6} - 601321y_{4n+6} + 1574628x_{2n+4} - 24052y_{2n+4} + 60]$
- $\frac{1}{360}[33y_{4n+5} - 3579y_{4n+4} + 132y_{2n+3} - 14316y_{2n+2} + 2160]$
- $\frac{1}{39600}[33y_{4n+6} - 393657y_{4n+4} + 132y_{2n+4} - 1574628y_{2n+2} + 237600]$
- $\frac{1}{360}[3579y_{4n+6} - 393657y_{4n+5} + 14316y_{2n+4} - 1574628y_{2n+3} + 2160]$

6. Each of the following expressions represents a Quintic integer

- $\frac{1}{840}\begin{bmatrix}5467x_{5n+5} - 49x_{5n+6} + 27335x_{3n+3} - 245x_{3n+4} \\ +54670x_{n+1} - 490x_{n+2}\end{bmatrix}$
- $\frac{1}{92400}\begin{bmatrix}601321x_{5n+5} - 49x_{5n+7} + 300665x_{3n+3} - 245x_{3n+5} \\ +6013210x_{n+1} - 490x_{n+3}\end{bmatrix}$
- $\frac{1}{10}[33x_{5n+5} - 49y_{5n+5} + 165x_{3n+3} - 245y_{3n+3} + 330x_{n+1} - 490y_{n+1}]$
- $\frac{1}{550}\begin{bmatrix}3579x_{5n+5} - 49y_{5n+6} + 17895x_{3n+3} - 245y_{3n+4} + 35790x_{n+1} \\ -490y_{n+2}\end{bmatrix}$
- $\frac{1}{60490}\begin{bmatrix}393657x_{5n+5} - 49y_{5n+7} + 1968285x_{3n+3} - 245y_{3n+5} \\ +3936570x_{n+1} - 490y_{n+3}\end{bmatrix}$
- $\frac{1}{840}\begin{bmatrix}601321x_{5n+6} - 5467x_{5n+7} + 3006605x_{3n+4} - 27335x_{3n+5} \\ +6013210x - 54670x_{n+3}\end{bmatrix}$
- $\frac{1}{550}\begin{bmatrix}33x_{5n+6} - 5467y_{5n+5} + 165x_{3n+4} - 27335y_{3n+3} + 330x_{n+2} \\ -5460y_{n+1}\end{bmatrix}$
- $\frac{1}{10}\begin{bmatrix}3579x_{5n+6} - 5467y_{5n+6} + 17895x_{3n+4} - 27335y_{3n+4} + 35790x_{n+2} \\ -54670y_{n+2}\end{bmatrix}$
- $\frac{1}{550}\begin{bmatrix}393657x_{5n+6} - 5467y_{5n+7} + 1968285x_{3n+4} - 27335y_{3n+5} \\ +3936570x_{n+2} - 54670y_{n+3}\end{bmatrix}$

- $\frac{1}{60490}\begin{bmatrix}33x_{5n+7}-601321y_{5n+5}+165x_{3n+5}-3006605y_{3n+3}+330x_{n+3}\\-6013210y_{n+1}\end{bmatrix}$
- $\frac{1}{550}\begin{bmatrix}3579x_{5n+7}-601321y_{5n+6}+17895x_{3n+5}-3006605y_{3n+4}+35790x_{n+3}\\-6013210y_{n+2}\end{bmatrix}$
- $\frac{1}{10}\begin{bmatrix}393657x_{5n+7}-601321y_{5n+7}+1968285x_{3n+5}-3006605y_{3n+5}\\+3936570x_{n+3}-6013210y_{n+3}\end{bmatrix}$
- $\frac{1}{360}\begin{bmatrix}33y_{5n+6}-3579y_{5n+5}+165y_{3n+4}-17895y_{3n+3}+330y_{n+2}\\-35790y_{n+1}\end{bmatrix}$
- $\frac{1}{39600}\begin{bmatrix}33y_{5n+7}-393657y_{5n+5}+165y_{3n+5}-1968285y_{3n+3}+330y_{n+3}\\-3936570y_{n+1}\end{bmatrix}$
- $\frac{1}{360}\begin{bmatrix}3579y_{5n+7}-393657y_{5n+6}+17895y_{3n+5}-1968285y_{3n+4}\\+35790y_{n+3}-3936570y_{n+2}\end{bmatrix}$

7. Remarkable observations

i) Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbola which are presented in the Table:2 below:

Table 2: Hyperbolas

S. No	Hyperbolas	(X, Y)
1	$X^2-21Y^2=2822400$	$(5467x_{n+1}-49x_{n+2}, 11x_{n+2}-1193x_{n+1})$
2	$X^2-21Y^2=34151040000$	$(501321x_{n+1}-49x_{n+3}, 11x_{n+3}-131219x_{n+1})$
3	$X^2-21Y^2=400$	$(33x_{n+1}-49y_{n+1}, 11y_{n+1}-7x_{n+1})$
4	$X^2-21Y^2=1210000$	$(3579x_{n+1}-49y_{n+2}, 11y_{n+2}-781x_{n+1})$
5	$X^2-21Y^2=14636160400$	$(393657x_{n+1}-49y_{n+3}, 11y_{n+3}-85903x_{n+1})$
6	$X^2-21Y^2=2822400$	$(601321x_{n+2}-5467x_{n+3}, 1193x_{n+3}-131219x_{n+2})$
7	$X^2-21Y^2=1210000$	$(33x_{n+2}-5467y_{n+1}, 1193y_{n+1}-7x_{n+2})$
8	$X^2-21Y^2=400$	$(3579x_{n+2}-5467y_{n+2}, 1193y_{n+2}-781x_{n+2})$
9	$X^2-21Y^2=1210000$	$(393657x_{n+2}-5467y_{n+3}, 1193y_{n+3}-85903x_{n+2})$
10	$X^2-21Y^2=14636160400$	$(33x_{n+3}-601321y_{n+1}, 131219y_{n+1}-7x_{n+3})$
11	$X^2-21Y^2=1210000$	$(3579x_{n+3}-601321y_{n+2}, 131219y_{n+2}-781x_{n+3})$
12	$X^2-21Y^2=400$	$(393657x_{n+3}-601321y_{n+3}, 11219y_{n+3}-85903x_{n+3})$
13	$X^2-21Y^2=518400$	$(33y_{n+2}-3579y_{n+1}, 781y_{n+1}-7y_{n+2})$
14	$X^2-21Y^2=6272640000$	$(33y_{n+3}-393657y_{n+1}, 85903y_{n+1}-7y_{n+3})$
15	$X^2-21Y^2=518400$	$(3579y_{n+3}-393657y_{n+2}, 85903y_{n+2}-781y_{n+3})$

ii) Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of parabolas which are presented in the Table:3 below:

Table 3: Parabolas

S. No	Parabolas	(X, Y)
1	$840X-21Y^2=1411200$	$(5467x_{2n+2}-49x_{2n+3}, 11x_{n+2}-1193x_{n+2})$
2	$92400X-21Y^2=17075520000$	$(601321x_{2n+2}-49x_{2n+4}, 11x_{n+3}-131219x_{n+1})$
3	$10X-21Y^2=200$	$(33x_{2n+2}-49y_{2n+2}, 11y_{n+1}-7x_{n+1})$
4	$550X-21Y^2=605000$	$(3579x_{2n+2}-49y_{2n+3}, 11y_{n+2}-781x_{n+1})$
5	$60490X-21Y^2=7318080200$	$(393657x_{2n+2}-49y_{2n+4}, 11y_{n+3}-85903x_{n+1})$
6	$840X-21Y^2=1411200$	$(601321x_{2n+3}-5467x_{2n+4}, 1193x_{n+3}-131219x_{n+2})$
7	$550X-21Y^2=605000$	$(33x_{2n+3}-5467y_{2n+2}, 1193y_{n+1}-7x_{n+2})$
8	$10X-21Y^2=200$	$(3579x_{2n+3}-5467y_{2n+3}, 1193y_{n+2}-781x_{n+2})$
9	$550X-21Y^2=605000$	$(393657x_{2n+3}-5467y_{2n+4}, 1193y_{n+3}-85903x_{n+2})$
10	$60490X-21Y^2=7318080200$	$(33x_{2n+4}-601321y_{2n+2}, 131219y_{n+1}-7x_{n+3})$
11	$550X-21Y^2=605000$	$(3549x_{2n+4}-601321y_{2n+3}, 131219y_{n+2}-781x_{n+3})$
12	$10X-21Y^2=200$	$(393657x_{2n+4}-601321y_{2n+4}, 131219y_{n+3}-85903x_{n+3})$
13	$360X-21Y^2=259200$	$(33y_{2n+3}-3579y_{2n+2}, 781y_{n+1}-7y_{n+2})$
14	$39600X-21Y^2=3136320000$	$(33y_{2n+4}-393657y_{2n+2}, 85903y_{n+1}-7y_{n+3})$
15	$360X-21Y^2=259200$	$(3579y_{2n+4}-393657y_{2n+3}, 85903y_{n+2}-781y_{n+3})$

Conclusion

In this paper, we have presented infinitely many integer solutions for the pellian like equation $3x^2-7y^2=20$. As the binary quadratic Diophantine equations are rich in variety, one may search for the other choices of pell equations and determine their integer solutions along with suitable properties.

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