International Journal of Advanced Science and Research
ISSN: 2455-4227; Impact Factor: RJIF 5.12
Received: 30-01-2019; Accepted: 01-03-2019
www.allsciencejournal.com
Volume 4; Issue 3; May 2019; Page No. 07-14

A peer search on $2x^2 - 7y^2 = 65$

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Abstract
The binary quadratic equation represented by the pellian like equation $2x^2 - 7y^2 = 65$ is analyzed for its distinct integer solutions. A few interesting relations among the solutions are given. Employing the solutions of the above hyperbola, we have obtained solutions of other choices of hyperbolas and parabolas.

Keywords: binary quadratic, hyperbola, parabola, Pell equation, integer solutions

Introduction

The binary quadratic Diophantine equation of the form $ax^2 - by^2 = N$, $(a, b, N \neq 0)$ are rich in variety and have been analyzed by many mathematicians for their respective integer solutions for particular values of a, b and N. In this context, one may refer [1-15].

This communication concerns with the problem of obtaining non-zero distinct integer solutions to the binary quadratic equation given by $2x^2 - 7y^2 = 65$ representing hyperbola. A few interesting relations among its solutions are presented. Knowing an integral solution of the given hyperbola, integer solutions for other choices of hyperbolas and parabolas are presented.

Method of Analysis

The Diophantine Equation representing the binary quadratic equation to be solved for its non-zero distinct integral solution is

$$2x^2 - 7y^2 = 65 \tag{1}$$

Consider the linear transformations

$$x = X + 7T \quad , \quad y = X + 2T \tag{2}$$

From (1) and (2), we have

$$X^2 = 14T^2 - 13 \tag{3}$$

whose smallest positive integer solution is

$$X_0 = 1 \quad T_0 = 1$$

To obtain the other solutions of (3), consider the Pellian equation is

$$X^2 = 14T^2 + 1 \tag{4}$$

whose smallest positive integer solution is

$$\tilde{X}_0 = 15 \quad \tilde{T}_0 = 4$$

The general solutions of (4) is given by

$$\tilde{X}_n = \frac{1}{2} f_n \, , \, \tilde{T}_n = \frac{1}{2\sqrt{14}} g_n$$

where

$$f_n = \left(15+4\sqrt{14}\right)^{n+1} + \left(15-4\sqrt{14}\right)^{n+1}$$

$$g_n = \left(15+4\sqrt{14}\right)^{n+1} - \left(15-4\sqrt{14}\right)^{n+1}$$

Applying Brahmagupta lemma between (X_0, T_0) and $(\tilde{X}_n, \tilde{T}_n)$, the other integer solutions of (3) are given by

$$\left.\begin{aligned} X_{n+1} &= \frac{f_n}{2} + \frac{\sqrt{14}}{2} g_n \\ T_{n+1} &= \frac{f_n}{2} + \frac{g_n}{2\sqrt{14}} \end{aligned}\right\} \quad (5)$$

From (2), (4) and (5) the values of x and y satisfying is given by

$$x_{n+1} = 4 f_n + \frac{21}{2\sqrt{14}} g_n$$

$$y_{n+1} = \frac{3}{2} f_n + \frac{8}{\sqrt{14}} g_n$$

The recurrence relations satisfied by x and y are given by

$$x_{n+3} - 30x_{n+2} + x_{n+1} = 0$$

$$y_{n+3} - 30y_{n+2} + y_{n+1} = 0$$

Some numerical examples of x_{n+1} and y_{n+1} satisfying (1) are given in the Table: 1 below

Table 1: Numerical Examples

n	x_{n+1}	y_{n+1}
-1	8	3
0	204	109
1	6112	3267
2	183156	97901
3	5488568	2933763
4	164473884	87914989

From the above table, we observe some interesting relations among the solutions which are presented below:

1. x_{n+1} is always even and y_{n+1} is always odd.

2. Relations among the solutions

- $x_{n+3} - 30x_{n+2} + x_{n+1} = 0$
- $28y_{n+1} - x_{n+2} + 15x_{n+1} = 0$
- $28y_{n+2} - 15x_{n+2} + x_{n+1} = 0$
- $28y_{n+3} - 449x_{n+2} + 15x_{n+1} = 0$
- $30x_{n+2} - x_{n+1} - x_{n+3} = 0$
- $840y_{n+1} - x_{n+3} + 449x_{n+1} = 0$

- $840y_{n+3} + x_{n+1} - 449x_{n+3} = 0$
- $x_{n+2} - 15x_{n+1} - 28y_{n+1} = 0$
- $x_{n+3} - 449x_{n+1} - 840y_{n+1} = 0$
- $y_{n+3} - 240x_{n+1} - 449y_{n+1} = 0$
- $y_{n+2} - 8x_{n+1} - 15y_{n+1} = 0$
- $56y_{n+2} - x_{n+3} + x_{n+1} = 0$
- $15y_{n+3} - 8x_{n+1} - 449y_{n+2} = 0$
- $449x_{n+3} - x_{n+1} - 840y_{n+3} = 0$
- $28y_{n+1} - 15x_{n+3} + 449x_{n+2} = 0$
- $28y_{n+2} - x_{n+3} + 15x_{n+2} = 0$
- $28y_{n+3} - 15x_{n+3} + x_{n+2} = 0$
- $15y_{n+2} - 8x_{n+2} + y_{n+1} = 0$
- $y_{n+3} - 16x_{n+2} - y_{n+1} = 0$
- $y_{n+3} - 8x_{n+2} - 15y_{n+2} = 0$
- $15y_{n+2} - y_{n+3} + 8x_{n+2} = 0$
- $449y_{n+2} - 8x_{n+3} + 15y_{n+1} = 0$
- $15y_{n+1} - 449y_{n+2} + 8x_{n+3} = 0$
- $15y_{n+3} - 8x_{n+3} - y_{n+2} = 0$
- $15x_{n+1} - x_{n+3} + 28y_{n+2} = 0$
- $y_{n+1} - 449y_{n+3} + 240x_{n+3} = 0$
- $y_{n+2} - 15y_{n+3} + 8x_{n+3} = 0$
- $y_{n+3} - 30y_{n+2} + y_{n+1} = 0$

3. Each of the following expression represents a Nasty Number

- $\frac{1}{9100}\left(45738x_{2n+2} - 42x_{2n+4} + 109200\right)$
- $\frac{6}{65}\left(32x_{2n+2} - 42y_{2n+2} + 130\right)$
- $\frac{2}{325}\left(816x_{2n+2} - 42y_{2n+3} + 1950\right)$
- $\frac{6}{29185}\left(24448x_{2n+2} - 42y_{2n+4} + 58370\right)$
- $\frac{3}{910}\left(45738x_{2n+3} - 1526x_{2n+4} + 3640\right)$

- $\frac{2}{325}\left(32x_{2n+3}-1526y_{2n+2}+1950\right)$
- $\frac{6}{65}\left(816x_{2n+3}-1526y_{2n+3}+130\right)$
- $\frac{2}{325}\left(2448x_{2n+3}-1526y_{2n+4}+1950\right)$
- $\frac{6}{29185}\left(32x_{2n+4}-45738y_{2n+2}+58370\right)$
- $\frac{2}{325}\left(816x_{2n+4}-45738y_{2n+3}+1950\right)$
- $\frac{6}{65}\left(24448x_{2n+4}-45738y_{2n+4}+130\right)$
- $\frac{3}{260}\left(32y_{2n+3}-816y_{2n+2}+1040\right)$
- $\frac{1}{2600}\left(32y_{2n+4}-24448y_{2n+2}+31200\right)$
- $\frac{3}{260}\left(816y_{2n+4}-2448y_{2n+3}+1040\right)$
- $\frac{3}{910}\left(1526x_{2n+2}-42x_{2n+3}+3640\right)$

4. Each of the following expressions represents a Cubical integer

- $\frac{1}{1820}\left[1526x_{3n+3}-42x_{3n+4}+4578x_{n+1}-126x_{n+2}\right]$
- $\frac{1}{54600}\left[45738x_{3n+3}-42x_{3n+5}+137214x_{n+1}-126x_{n+3}\right]$
- $\frac{1}{65}\left[32x_{3n+3}-42y_{3n+3}+96x_{n+1}-126y_{n+1}\right]$
- $\frac{1}{975}\left[816x_{3n+3}-42y_{3n+4}+2448x_{n+1}-126y_{n+2}\right]$
- $\frac{1}{29185}\left[24448x_{3n+3}-42y_{3n+5}+73344x_{n+1}-126y_{n+3}\right]$
- $\frac{1}{975}\left[32x_{3n+4}-1526y_{3n+3}+96x_{n+2}-4578y_{n+1}\right]$
- $\frac{1}{65}\left[816x_{3n+4}-1526y_{3n+4}+2448x_{n+2}-4578y_{n+2}\right]$
- $\frac{1}{975}\left[24448x_{3n+4}-1526y_{3n+5}+73344x_{n+2}-44578y_{n+3}\right]$
- $\frac{1}{29185}\left[32x_{3n+5}-45738y_{3n+3}+96x_{n+3}-137214y_{n+1}\right]$

- $\frac{1}{975}\left[816x_{3n+5}-45738y_{3n+4}+2448x_{n+3}-137214y_{n+2}\right]$
- $\frac{1}{29185}\left[24448x_{3n+3}-42y_{3n+5}+73344x_{n+1}-126y_{n+3}\right]$
- $\frac{1}{520}\left[32y_{3n+4}-816y_{3n+3}+96y_{n+2}-2448y_{n+1}\right]$
- $\frac{1}{15600}\left[32y_{3n+5}-24448y_{3n+3}+96y_{n+3}-73344y_{n+1}\right]$
- $\frac{1}{520}\left[816y_{3n+5}-24448y_{3n+4}+2448y_{n+3}-73344y_{n+2}\right]$
- $\frac{1}{1820}\left[45738x_{3n+4}-1526x_{3n+5}+137214x_{n+2}-4578x_{n+3}\right]$

5. Each of the following expressions represents a Bi-quadratic integer:

- $\frac{1}{1820}\left[1526x_{4n+4}-42x_{4n+5}+6104x_{2n+2}-168x_{2n+3}+10920\right]$
- $\frac{1}{54600}\left[45738x_{4n+4}-42x_{4n+6}+182952x_{2n+2}-168x_{2n+4}+327600\right]$
- $\frac{1}{65}\left[32x_{4n+4}-42y_{4n+4}+128x_{2n+2}-168y_{2n+2}+390\right]$
- $\frac{1}{975}\left[816x_{4n+4}-42y_{4n+5}+3264x_{2n+2}-168y_{2n+3}+5850\right]$
- $\frac{1}{29185}\left[24448x_{4n+4}-42y_{4n+6}+97792x_{n+1}-168y_{n+3}+175110\right]$
- $\frac{1}{1820}\left[45738x_{4n+5}-1526x_{4n+6}+182952x_{2n+3}-6104x_{2n+4}+10920\right]$
- $\frac{1}{975}\left[32x_{4n+5}-1526y_{4n+4}+128x_{2n+3}-6104y_{2n+2}+5850\right]$
- $\frac{1}{65}\left[816x_{4n+5}-1526y_{4n+5}+3264x_{2n+3}-6104y_{2n+3}+390\right]$
- $\frac{1}{975}\left[24448x_{4n+5}-1526y_{4n+6}+97792x_{2n+3}-6104y_{2n+4}+5850\right]$
- $\frac{1}{29185}\left[32x_{4n+6}-45738y_{4n+4}+128x_{2n+4}-182952y_{2n+2}+175110\right]$
- $\frac{1}{975}\left[816x_{4n+6}-45738y_{4n+5}+3264x_{2n+4}-182952y_{2n+3}+5850\right]$
- $\frac{1}{65}\left[24448x_{4n+6}-45738y_{4n+6}+97792x_{2n+4}-182952y_{2n+4}+390\right]$
- $\frac{1}{15600}\left[32y_{4n+6}-24448y_{4n+4}+128y_{2n+4}-97792y_{2n+2}+93600\right]$

- $\frac{1}{520}\left[816y_{4n+6}-24448y_{4n+5}+3264y_{2n+4}-97792y_{2n+3}+3120\right]$
- $\frac{1}{520}\left(32y_{4n+5}-816y_{4n+4}+128y_{2n+3}-3264y_{2n+2}+3120\right)$

6. Each of the following expressions represents a Quintic integer

- $\frac{1}{1820}\left[1526x_{5n+5}-42x_{5n+6}+7630x_{3n+3}-210x_{3n+4}+15260x_{n+1}-420x_{n+2}\right]$
- $\frac{1}{54600}\left[45738x_{5n+5}-42x_{5n+7}+228690x_{3n+3}-210x_{3n+5}+457380x_{n+1}-420x_{n+3}\right]$
- $\frac{1}{65}\left[32x_{5n+5}-42y_{5n+5}+160x_{3n+3}-210y_{3n+3}+320x_{n+1}-420y_{n+1}\right]$
- $\frac{1}{975}\left[816x_{5n+5}-42y_{5n+6}+4080x_{3n+3}-210y_{3n+4}+8160x_{n+1}-420y_{n+2}\right]$
- $\frac{1}{29185}\left[24448x_{5n+5}-42y_{5n+7}+122240x_{3n+3}-210y_{3n+5}+244480x_{n+1}-420y_{n+3}\right]$
- $\frac{1}{1820}\left[45738x_{5n+6}-1526x_{5n+7}+228690x_{3n+4}-7630x_{3n+5}+457380x_{n+2}-15260x_{n+3}\right]$
- $\frac{1}{975}\left[32x_{5n+6}-1526y_{5n+5}+160x_{3n+4}-7630y_{3n+3}+320x_{n+2}-15260y_{n+1}\right]$
- $\frac{1}{65}\left[816x_{5n+6}-1526y_{5n+6}+4080x_{3n+4}-7630y_{3n+4}+8160x_{n+2}-15260y_{n+2}\right]$
- $\frac{1}{975}\left[24448x_{5n+6}-1526y_{5n+7}+122240x_{3n+4}-7630y_{3n+5}+244480x_{n+2}-15260y_{n+3}\right]$
- $\frac{1}{29185}\left[32x_{5n+7}-45738y_{5n+5}+160x_{3n+5}-228690y_{3n+3}+320x_{n+3}-457380y_{n+1}\right]$
- $\frac{1}{975}\left[816x_{5n+7}-45738y_{5n+6}+4080x_{3n+5}-228690y_{3n+4}+8160x_{n+3}-457380y_{n+2}\right]$
- $\frac{1}{65}\left[24448x_{5n+7}-45738y_{5n+7}+122240x_{3n+5}-228690y_{3n+5}+244480x_{n+3}-457380y_{n+3}\right]$
- $\frac{1}{520}\left[32y_{5n+6}-816y_{5n+5}+160y_{3n+4}-4080y_{3n+3}+320y_{n+2}-8160y_{n+1}\right]$
- $\frac{1}{15600}\left[32y_{5n+7}-24448y_{5n+5}+160y_{3n+5}-122240y_{3n+3}+320y_{n+3}-244480y_{n+1}\right]$

7. Remarkable observations

i) Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbola which are presented in Table: 2 below:

Table 2: Hyperbolas

S.NO	Hyperbolas	(X, Y)
1	$X^2 - 14Y^2 = 13249600$	$(1526x_{n+1} - 42x_{n+2}, 16x_{n+2} - 408x_{n+1})$
2	$X^2 - 14Y^2 = 218400$	$(45738x_{n+1} - 42x_{n+3}, 16x_{n+3} - 12224x_{n+1})$
3	$X^2 - 14Y^2 = 16900$	$(32x_{n+1} - 42y_{n+1}, 16y_{n+1} - 6x_{n+1})$
4	$X^2 - 14Y^2 = 3802500$	$(816x_{n+1} - 42y_{n+2}, 16y_{n+2} - 218x_{n+1})$
5	$X^2 - 14Y^2 = 3407056900$	$(24448x_{n+1} - 42y_{n+3}, 16y_{n+3} - 6534x_{n+1})$
6	$X^2 - 14Y^2 = 7280$	$(45738x_{n+2} - 1526x_{n+3}, 408x_{n+3} - 12224x_{n+2})$
7	$X^2 - 14Y^2 = 3802500$	$(32x_{n+2} - 1526y_{n+1}, 408y_{n+1} - 6x_{n+2})$
8	$X^2 - 14Y^2 = 16900$	$(816x_{n+2} - 1526y_{n+2}, 408y_{n+2} - 218x_{n+2})$
9	$X^2 - 14Y^2 = 950625$	$(24448x_{n+2} - 1526y_{n+3}, 408y_{n+3} - 6534x_{n+2})$
10	$X^2 - 14Y^2 = 3407056900$	$(32x_{n+3} - 45738y_{n+1}, 12224y_{n+1} - 6x_{n+3})$
11	$X^2 - 14Y^2 = 16900$	$(24448x_{n+3} - 45738y_{n+3}, 12224y_{n+3} - 6534x_{n+3})$
12	$X^2 - 14Y^2 = 1081600$	$(32y_{n+2} - 816y_{n+1}, 218y_{n+1} - 6y_{n+2})$
13	$X^2 - 14Y^2 = 973440000$	$(32y_{n+3} - 24448y_{n+1}, 6534y_{n+1} - 6y_{n+3})$
14	$X^2 - 14Y^2 = 1081600$	$(816y_{n+3} - 24448y_{n+2}, 6534y_{n+2} - 218y_{n+3})$
15	$X^2 - 14Y^2 = 3802500$	$(816x_{n+3} - 45738y_{n+2} \rightleftarrows, 12224y_{n+2} - 218x_{n+3})$

ii) Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of parabola which are presented in Table: 3 below:

Table 3: Parabolas

S.no	Parabolas	(X, Y)
1	$1820X - 14Y^2 = 6624800$	$(1526x_{2n+2} - 42x_{2n+3}, 16x_{n+2} - 408x_{n+1})$
2	$54600X - 14Y^2 = 5962320000$	$(45738x_{2n+2} - 42x_{2n+4}, 16x_{n+3} - 12224x_{n+1})$
3	$65X - 14Y^2 = 8450$	$(32x_{2n+2} - 42y_{2n+2}, 16y_{n+1} - 6x_{n+1})$
4	$975X - 14Y^2 = 1901250$	$(816x_{2n+2} - 42y_{2n+3}, 16y_{n+2} - 218x_{n+1})$
5	$29185X - 14Y^2 = 1703528450$	$(24448x_{2n+2} - 42y_{2n+4}, 16y_{n+3} - 6534x_{n+1})$
6	$1820X - 14Y^2 = 6624800$	$(45738x_{2n+3} - 1526x_{2n+4}, 408x_{n+3} - 12224x_{n+2})$
7	$975X - 14Y^2 = 1901250$	$(32x_{2n+3} - 1526y_{2n+2}, 408y_{n+1} - 6x_{n+2})$
8	$65X - 14Y^2 = 8450$	$(816x_{2n+3} - 1526y_{2n+3}, 408y_{n+2} - 218x_{n+2})$
9	$975X - 14Y^2 = 1901250$	$(24448x_{2n+3} - 1526y_{2n+4}, 408y_{n+3} - 6534x_{n+2})$
10	$29185X - 14Y^2 = 1703528450$	$(32x_{2n+4} - 45738y_{2n+2}, 12224y_{n+1} - 6x_{n+3})$
11	$975X - 14Y^2 = 1901250$	$(816x_{2n+4} - 45738y_{2n+3}, 12224y_{n+2} - 218x_{n+3})$
12	$65X - 14Y^2 = 8450$	$(24448x_{2n+4} - 45738y_{2n+4}, 12224y_{n+3} - 6534x_{n+3})$
13	$520X - 14Y^2 = 540800$	$(32y_{2n+3} - 816y_{2n+2}, 218y_{n+1} - 6y_{n+2})$
14	$520X - 14Y^2 = 540800$	$(816y_{2n+4} - 24448y_{2n+3}, 6534y_{n+2} - 218y_{n+3})$
15	$15600X - 14Y^2 = 486720000$	$(32y_{2n+4} - 24448y_{2n+2}, 6534y_{n+1} - 6y_{n+3})$

Conclusion

In this paper, we have presented infinitely many integer solutions for the pellian like equation $2x^2 - 7y^2 = 65$. As the binary quadratic Diophantine equations are rich in variety, one may search for the other choices of pell equations and determine their integer solutions along with suitable properties.

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