



A survey on fuzzy entropy measures and their applications

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Abstract

The concept of fuzzy set was introduced by Zadeh [25] and this concept is now being applied to every branch of knowledge. In information theory and telecommunications, the entropy is the basic subject and it measures the fuzziness in fuzzy sets. In the theory and applications of the fuzzy sets, many research investigations have been made. For expressing the fuzziness of fuzzy sets, mathematically, fuzzy entropy has been used. The aim of the present paper is to carry out literature survey of different types of fuzzy information measures and their applications in diversified fields.

Keywords: entropies, fuzzy sets, fuzzy entropy functions

Introduction

The conceptual theory of fuzzy sets was firstly proposed by Zadeh [25] which solves the problems involving the indefiniteness. The indefiniteness usually arises due to some sort of intrinsic ambiguity which plays a significant role in the concept of fuzzy sets. Fuzzy sets theory deal with uncertainty and vagueness where the membership function is the key function to characterize the degree of belongingness of an element to a set. According to Zadeh [25] theory, the possibility to the degree of non-membership of the element to the fuzzy set and the degree of membership function is always equal to 1. Some definitions are given below:

Entropy

Some uncertainty is always there for each and every probability distribution. The entropy provides a quantitative measure of this uncertainty. Entropy is particularly important in information theory and was introduced there by Shannon [23].

Fuzzy Sets and Membership Function

Zadeh [25] defined the fuzzy set and membership function as under:

“Let X be a space of points (objects), with a generic element of X denoted by x . Thus, $X = \{x\}$. A fuzzy set(class) A in X is characterized by a membership (characteristic) function $f_A(x)$ which associates with each point in X a real number in the interval $[0,1]$, with the value of $f_A(x)$ at x representing the grade of membership of x in A . Thus, the nearer the value of $f_A(x)$ to unity, the higher the grade of membership of x in A . When A is a set in the ordinary sense of the term, its membership function can take on only two values 0 and 1, with $f_A(x) = 1$ or 0 according as x does or does not belong to A . Thus, in this case $f_A(x)$ reduces to the familiar characteristic function of a set A . (When there is a need to differentiate between such sets and fuzzy sets, the sets with two-valued characteristic functions will be referred to as ordinary sets or simply sets)”.

Review of Literature

Let A_1, A_2, \dots, A_n be the n possible outcomes of a random experiment with their respective probabilities P_1, P_2, \dots, P_n , giving rise to the probability distribution

$$P = (p_1, p_2, \dots, p_n); \sum_{i=1}^n p_i = 1; p_i \geq 0, \dots, p_n \geq 0$$

Some uncertainty always occurs when an experiment is performed. A brief review of literature is given as under: Shannon [23] entropy measure is defined as

$$H(P) = - \sum_{i=1}^n p_i \log p_i \tag{2.1}$$

Renyi [22] defined entropy of order α as

$$H_\alpha(P) = \frac{1}{1-\alpha} \log \left(\sum_{i=1}^n p_i^\alpha \right), \alpha \neq 1, \alpha > 0 \tag{2.2}$$

First non-negative entropy was introduced by Havrada and Charvat ^[7] as mentioned below:

$$H_{\alpha}(P) = \frac{\left[\sum_{i=1}^n p_i^{\alpha} \right]^{-1}}{2^{1-\alpha} - 1}, \alpha \neq 1, \alpha > 0. \quad (2.3)$$

Mathai and Rathie [19] has considered the following entropy measure:

$$H_{\alpha}(P) = \frac{1}{\alpha - 1} \left[\sum_{i=1}^n p_i^{2-\alpha} - 1 \right], \alpha \neq 1, -\infty < \alpha < 2. \quad (2.4)$$

Boekee and Lubbe [5] proposed the following R-norm information measure:

$$H_R(P) = \frac{R}{R-1} \left[1 - \left(\sum_{i=1}^{\infty} p_i^R \right)^{\frac{1}{R}} \right], R > 0, R \neq 1 \quad (2.5)$$

The above defined measure is called R-norm information measure and this measure is an extension of Shannon's entropy, i.e. when $R \rightarrow 1$, it tends to Shannon's entropy.

Kapur [14] introduced the generalized measures of entropy given by

$$H_{\beta}^{\alpha}(P) = \frac{1}{\alpha + \beta - 2} \left[\sum_{i=1}^n p_i^{\alpha} + \sum_{i=1}^n p_i^{\beta} - 2 \right], \alpha \geq 1, \beta \leq 1 \text{ or } \alpha \leq 1, \beta \geq 1 \quad (2.6)$$

Classical Shannon information entropy was analyzed by Pal and Pal ^[20] and they proposed an exponential entropy as mentioned below:

$$H(P) = \sum_{i=1}^n p(x_i) \left(e^{(1-p(x_i))} - 1 \right) \quad (2.7)$$

These authors conclude that the exponential entropy has an advantage over Shannon's entropy. For the uniform probability distribution $p = (\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n})$ exponential entropy has a fixed upper bound

$$\lim_{n \rightarrow \infty} H \left(\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n} \right) = e - 1$$

Which is not the case for Shannon's entropy.

Hooda and Ram [12] characterized a generalized R-norm information measure of degree β as given below:

$$H_R^{\beta}(P) = \frac{R}{R + \beta - 2} \left[1 - \left(\sum_{i=1}^{\infty} p_i^{\frac{R}{2-\beta}} \right)^{\frac{2-\beta}{R}} \right], 0 < \beta \leq 1, R > 0, R + \beta \neq 2 \quad (2.8)$$

Kvalseth ^[17] defined generalized exponential entropy of order α as

$$H_{\alpha}(P) = \frac{\sum_{i=1}^n p(x_i) \left(e^{1-p(x_i)} - 1 \right)}{\alpha}, \alpha > 0 \quad (2.9)$$

Hooda and Sharma ^[13] generalized measure given by Hooda and Ram ^[12] and characterized the following R-norm

$$H_R^{(\alpha, \beta)}(P) = \frac{R}{R - 2\alpha + \beta} \left[1 - \left(\sum_{i=1}^{\infty} p_i^{\frac{R}{2\alpha - \beta}} \right)^{\frac{2\alpha - \beta}{R}} \right], \alpha \geq 1, 0 < \beta \leq 1, R(> 0) \neq 1, R + \beta \neq 2\alpha \quad (2.10)$$

Bhat and Baig ^[3] defined an entropy of order α and type β as

$$H_{\beta}^{\alpha}(P) = \frac{\beta}{1-\alpha} \log_D [p_i^{\alpha\beta}], 0 < \alpha < 1, \alpha < \beta \leq 1. \quad (2.11)$$

Fuzzy Entropy

Hooda and Raich [10] discussed the fuzzy entropy as:

“The measure of uncertainty is adopted as a measure of information. Hence, the measure of fuzziness is known as fuzzy information measures. The measure of the amount of fuzzy information obtained from a fuzzy set or fuzzy system is known as fuzzy entropy”.

To measure the amount of fuzzy information a set of four properties was proposed by De Luca and Termini [18] and now as a criterion, these properties are widely acceptable and used to create any new fuzzy entropy. The fuzzy set theory involves the entropy which measures the quantitative average of ambiguity or any difficulty in deciding whether an element belongs to a set or not. Therefore, Hooda and Raich [10] gave a new definition on the lines of De Luca and Termini [18] as below:

“A real function $H: \text{FSs}(X) \rightarrow [0,1]$ is called a fuzzy information measure on $\text{FSs}(X)$ if it satisfies the following properties:

1. Sharpness: $H(A)$ is minimum if and only if A is a crisp set, i.e. $\mu_A(x) = 0$ or $1: \forall x$.
2. Maximality: $H(A)$ is maximum if and only if A is most fuzzy sets, i.e. $\mu_A(x) = 0.5; \forall x$.
3. Resolution: $H(A) \geq H(A^*)$, where A^* is a sharpened version of A .
4. Symmetry: $H(A) = H(\bar{A})$, where \bar{A} is complement of A , i.e. $\mu_{\bar{A}}(x_i) = 1 - \mu_A(x_i)$.

Here $\mu_A(x)$ and $1 - \mu_A(x)$ gives the same degree of fuzziness. On the basis of this, De Luca and Termini [18] proposed the following measure of fuzzy entropy corresponding to Shannon [23] entropy:

$$H(A) = - \sum_{i=1}^n [\mu_A(x_i) \log \mu_A(x_i) + (1 - \mu_A(x_i)) \log(1 - \mu_A(x_i))] \quad (3.1)$$

Corresponding to Renyi's [22] entropy of order α (given in (2.2)), Bhandari and Pal [2] has taken the following fuzzy entropy measure of order α :

$$H_{\alpha}(A) = \frac{1}{1-\alpha} \sum_{i=1}^n \log [\mu_A^{\alpha}(x_i) + (1 - \mu_A(x_i))^{\alpha}], \alpha \neq 1, \alpha > 0 \quad (3.2)$$

Kapur[15] has considered the fuzzy entropy corresponding to Havrada and Charvat [7] entropy given in (2.3) as

$$H_{\alpha}(A) = \frac{1}{1-\alpha} \sum_{i=1}^n [\{\mu_A^{\alpha}(x_i) + (1 - \mu_A(x_i))^{\alpha}\} - 1], \alpha \neq 1, \alpha > 0 \quad (3.3)$$

Baig et. al. [1] gave the following measure of fuzzy entropy corresponding to the entropy given by Mathai and Rathie [19] as

$$H_{\alpha}(A) = \frac{1}{\alpha-1} \left[\sum_{i=1}^n \{\mu_A^{2-\alpha}(x_i) + (1 - \mu_A(x_i))^{2-\alpha}\} - 1 \right] \quad (3.4)$$

Corresponding to the measure given by Boekee and Lubbe [5] in (2.5), Hooda [8] considered the R-norm fuzzy entropy measure given by

$$H_R(A) = \frac{R}{R-1} \sum_{i=1}^n \left[1 - (\mu_A^R(x_i) + (1 - \mu_A(x_i))^R)^{\frac{1}{R}} \right], R > 0, R \neq 1 \quad (3.5)$$

Kapur [16] defined the fuzzy entropy corresponding to Kapur [14] generalized measures of entropy given in (2.6) as

$$H_{\alpha,\beta}(A) = \frac{1}{\beta-\alpha} \sum_{i=1}^n [\{\mu_A^{\alpha}(x_i) + (1 - \mu_A(x_i))^{\alpha}\} - \{\mu_A^{\beta}(x_i) + (1 - \mu_A(x_i))^{\beta}\}], \quad (3.6)$$

$\alpha \geq 1, \beta \leq 1$ or $\alpha \leq 1, \beta \geq 1$

Pal and Pal ^[21] gave the exponential fuzzy entropy corresponding to the entropy given by Pal and Pal ^[20] in (2.7) as

$$H(A) = \frac{1}{n(\sqrt{e}-1)} \sum_{i=1}^n \left[\mu_A(x_i) e^{(1-\mu_A(x_i))} + (1-\mu_A(x_i)) e^{\mu_A(x_i)} - 1 \right] \quad (3.7)$$

Corresponding to the measure given by Hooda and Ram ^[12] in (2.8), Hooda and Bajaj ^[9] defined a generalized measure of R-norm fuzzy information measure of degree β as given by

$$H_R^\beta(P) = \frac{R}{R+\beta-2} \sum_{i=1}^n \left[1 - \left(\mu_A^{\frac{R}{2-\beta}}(x_i) + (1-\mu_A(x_i))^{\frac{R}{2-\beta}} \right)^{\frac{2-\beta}{R}} \right], \quad 0 < \beta \leq 1, R > 0 (\neq 1), R + \beta \neq 2 \quad (3.8)$$

Verma and Sharma ^[24] defined the exponential fuzzy entropy of order α corresponding to the entropy given by Kvalseth ^[17] given in (2.9) as

$$H_\alpha(A) = \frac{1}{n(e^{(1-0.5^\alpha)}-1)} \sum_{i=1}^n \left[\mu_A(x_i) e^{(1-\mu_A^\alpha(x_i))} + (1-\mu_A(x_i)) e^{(1-(1-\mu_A(x_i))^\alpha)} - 1 \right], \quad \alpha > 0 \quad (3.9)$$

Corresponding to the measure given by Hooda and Sharma ^[13] given in (2.10), Hooda and Jain ^[11] has taken the following R-norm fuzzy information measure:

$$H_{R}^{\alpha,\beta}(A) = \frac{R}{R-2\alpha+\beta} \sum_{i=1}^n \left[1 - \left\{ \mu_A^{\frac{R}{2\alpha-\beta}}(x_i) + (1-\mu_A(x_i))^{\frac{R}{2\alpha-\beta}} \right\}^{\frac{2\alpha-\beta}{R}} \right], \quad \alpha \geq 1, 0 < \beta \leq 1, R(>0) \neq 1, R + \beta \neq 2\alpha \quad (3.10)$$

Gupta et. al. ^[6] defined α -exponential fuzzy entropy as

$$H_\alpha(A) = \frac{1}{n(2^{1-\alpha} e^{1-2^\alpha} - 1)} \sum_{i=1}^n \left[\mu_A^\alpha(x_i) e^{1-\mu_A^\alpha(x_i)} + (1-\mu_A(x_i))^\alpha e^{1-(1-\mu_A(x_i))^\alpha} - 1 \right], \quad 0 < \alpha \leq 1 \quad (3.11)$$

When $\alpha=1$, the above measure reduces to Pal and Pal ^[21] fuzzy entropy measure.

Analogous to the measure given by Bhat and Baig ^[3] given in (2.11), Bhat and Baig ^[4] defined a new generalized fuzzy information measure of order α and type β as

$$H_{\alpha}^{\beta}(A) = \frac{\beta}{\alpha} \log_D \left[\frac{1}{n} \sum_{i=1}^n \mu_A^{\alpha\beta}(x_i) + (1-\mu_A(x_i))^{\alpha\beta} \right], \quad 0 < \alpha < 1, 0 < \beta \leq 1 \quad (3.12)$$

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