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Complex integrals - Methods and their applicability

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Abstract

This paper reviews the methods to evaluate different types of complex integrals. To evaluate the complex integrals in an easy way, there are various properties of complex integration like Cauchy's Integral formula, the Argument Principle, Cauchy's Residue Theorem etc. The various conditions and methods to apply on complex integrals are discussed in this paper. Only the correct use of these properties can give us valid answers.

Keywords: Contour, analytic, residue, poles, meromorphic, regular, rectifiable

1. Introduction

As in real variable, a distinction is made between definite and indefinite integrals- the former being regarded as the limit of sum and the latter as a process inverse to differentiation. Similar is the distinction between definite and indefinite integrals of a complex variable. Definite integrals of complex variable are known as line integrals and indefinite integral of a complex variable is a function whose derivative equals a given analytic function in a region [8].

1.1 Contours

Contours are the set of curves on which we identify contour integration. A contour is a directed curve which is made up of a finite sequence of directed smooth curves whose endpoints are coordinated to give a single direction.

1.2 Contour Integrals

The contour integral of a complex function f: C → C is a overview of the integral for real-valued functions. The integral $\int_{(0,0)}^{(1,1)} z^2$ dz along the curve y = x^2 is called the line integral of the complex function f (z) along the curve C and is denoted by $\int_c$ f(z)dz. In case the end points of the curve are coincide so that C is a closed curve then this integral is called contour integral and is denoted by $\oint_C$ f(z)dz.

2. Different methods for evaluating the complex integrals

2.1 line integrals: (direct evaluation)

2.1.1 Applicability

A line integral is an integral where the function to be integrated is evaluated along a line or a curve. The terms path integral, curve integral, contour integral and curvilinear integral are also used for line integrals in the complex plane [10].

Example 1: (Line Integral along a curve i.e. circle)

The value of I = $\int \frac{dz}{z-a}$ over L where L represents a circle |z-a|= r is 2πi

Here parametric equation of the circle | z-a|= r is z= a+r$e^{i\theta}$; 0≤$\theta \leq 2\pi$

$$dz = ire^{i\theta}$$

$$\therefore \quad I = \int_0^{2\pi} \frac{ire^{i\theta}}{re^{i\theta}}\, d\theta = 2\pi i$$

Example 2: (Integral along a straight line)

The value of I = $\int z^2$ dz along the straight line OM where O is origin and M is the point z= 3+i is 6+$\frac{26}{3}$i

Here equation of OM is x=3y

$$dx = 3dy$$

$$z = x+iy \Rightarrow dz = dx+idy$$

$$z^2 = x^2 - y^2 + 2xyi$$

$$\therefore \quad I = \int z^2\, dz$$

$$= \int (x^2 - y^2 + 2xyi)(dx+idy)$$

$$= \int_0^1 (9y^2 - y^2 + 6y^2 i)(3dy + idy)$$

$$= \int_0^1 y^2(8+6i)(3+i)dy$$

$$= (8+6i)(3+i)[\frac{y^3}{3}] = 6+\frac{26}{3}i$$

2.2 Darboux property

If f (z) is continuous on a contour L of length l and |f (z)| ≤ M then

$$|\int_L f(z)dz\,| \leq Ml$$

2.2.1 Applicability

This property can be used when the length of the curve and maximum value of the function to be integrated is known or can be found easily [11].

Example 3: The value of $|\int_C \frac{1}{z^2}dz| \leq 2$ where C is the line segment joining -1+i and 1+i

From given statement, Length of the curve and Max value of f(z) can be found

Here $\quad 1 \leq z \leq \sqrt{2}$

$$\therefore \quad \frac{1}{2} \leq \frac{1}{|z|^2} \leq 1$$

Thus M= 1 and arc length L=2

$$\therefore \quad |\int_C \frac{1}{z^2}dz| \leq ML = 2$$

2.3 Cauchy's goursat theorem

If f(z) is analytic in a simply connected domain D and C be any closed continuous rectifiable curve in D then $\int_C f(z)dz = 0$

2.3.1 Applicability
This property can be used when the function to be integrated is analytic and the contour is closed and continuous [9].

Example 4: The value of $\oint_C z^2 dz$ = 0, where C is the contour |z| = 1.
Given integral is analytic and C being a circle is closed.
Here f (z) = z^2 is analytic and C is a simple closed contour.
Hence by Cauchy–Goursat theorem

$$\oint_C z^2 dz = 0$$

2.4 Cauchy's Integral Formula
If f(z) is analytic function in a simply connected domain D enclosed by a rectifiable Jordan curve C and f(z) is continuous on C then f(z_0) = $\frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}$ dz ; z_0 is any point of D [6].

2.4.1 Applicability
This formula is used when the given function is of the type $\frac{f(z)}{(z-z_0)(z-z_1).....(z-z_n)}$ where c_0, c_1, ----, c_n are closed contours inside C and C_0 encircles the point z_0 , C_1 encircles the point z_1, ------, C_n encircles the point z_n. Also f (z) should be analytic [1].

Example 5: $\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}$ dz = 4πi where C is the circle |z-i| = 3

Given integral $\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}$ = of the form $\frac{f(z)}{(z-z_0)(z-z_1).....(z-z_n)}$.

Here $\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}$ dz

$= \int_{C_1} \frac{\sin \pi z^2 + \cos \pi z^2/(z-2)}{z-1}$ dz+$\int_{C_2} \frac{\sin \pi z^2 + \cos \pi z^2/(z-1)}{z-2}$ dz

Where C_1 and C_2 are closed contours inside C, C_1 encircles the point z=1 and C_2 encircles the point z=2

For $\int_{C_1} \frac{\sin \pi z^2 + \cos \pi z^2/(z-2)}{z-1}$ dz, Take f(z)

$= \sin \pi z^2 + \cos \pi z^2/(z-2)$

∴ $\int_{C_1} \frac{f(z)}{z-1}$ dz = 2πi f(1) = 2πi

And for $\int_{C_2} \frac{\sin \pi z^2 + \cos \pi z^2/(z-1)}{z-2}$ dz, Take f(z)

$= \sin \pi z^2 + \cos \pi z^2/(z-1)$

∴ $\int_{C_2} \frac{f(z)}{z-2}$ dz = 2πi f(2) = 2πi

Hence $\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}$ dz = 2πi+2πi= 4πi

2.5 Derivative of analytic function
If f(z) is analytic within and on the boundary C of a simply connected region D and let z_0 be any point with in C then

$$f'(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^2} dz$$

Generalization: $f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}$ dz

2.5.1 Applicability
This property is used when the function to be integrated is of the form $\frac{f(z)}{(z-z_0)^{n+1}}$. Also f (z) should be analytic [4].

Example 6: The value of $\int_C \frac{e^{2z}}{(z+1)^4}$ dz = 8πi$\frac{e^{-2}}{3}$ where C is the circle |z| = 3

Given integral = $\int_C \frac{e^{2z}}{(z+1)^4}$ of the form $\frac{f(z)}{(z-z_0)^{n+1}}$

Here f(z)= e^{2z} and z_0 = -1

Using $f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}$ dz

Take n=3, $f'''(-1) = 8e^{-2}$

∴ $8e^{-2} = \frac{3!}{2\pi i}\int_C \frac{e^{2z}}{(z+1)^4}$ dz

Hence $\int_C \frac{e^{2z}}{(z+1)^4}$ dz = 8πi$\frac{e^{-2}}{3}$

2.6 The argument principle
If f(z) is meromorphic inside a closed contour C and has no zeros on C then $\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}$ dz = N-P where N is number of zeros and P is number of poles of f(z) inside C [5].

2.6.1 Pole
If f(z) = $\frac{g(z)}{(z-a)^m}$ is any function . Then the function f(z) has a pole of order m at z = a .

2.6.2 Applicability
This principle is used when the function to be integrated is of the form $\frac{f'(z)}{f(z)}$.

Example 7: The value of $\int_C \frac{2z+1}{z^2+z}$ dz = 4πi where C is the circle |z| = 2

Given integral $\int_C \frac{2z+1}{z^2+z}$ dz = of the form $\frac{f'(z)}{f(z)}$.

Here f (z) = $z^2 + z$, $f'(z)$ = 2z+1
Number of zeros = 2
And number of poles = 0

Hence $\frac{1}{2\pi i}\int_C \frac{2z+1}{z^2+z}$ dz = 2πi(2-0)= 4πi

2.7 Cauchy's residue theorem
If f(z) is regular except at finite number of poles with in a closed contour C and continuous on boundary C then $\int_C f(z)dz$ = 2πi $\sum R$ where $\sum R$ = sum of residues of f(z) at its poles with in C [7].

2.7.1 Methods of finding the residues
i) If f(z) has a simple pole (pole of order 1) at z = a, then
Res {f(z)} = $\lim_{z\to a}(z-a)f(z)$.

ii) If f(z) has a pole of order m at z = a then
Res {f(z)} = $\frac{1}{(m-1)!}\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}[(z-a)^m f(z)]$.

iii) Residue of f(z) at z = a pole (simple or order m) = Coefficient of 1/z in the expansion of f(z)

iv) Residue of f(z) at z = ∞ = $\lim_{z\to\infty}[-zf(z)]$.
Or = - [coefficient of 1/z in the expansion of f(z).

2.7.2 Applicability
This principle is used when the function to be integrated is regular except at finite number of poles [2].

Example 8: The value of $\int_C \frac{2z+1}{z^2+z}$ dz = 4πi where C is the circle |z| = 2

Given integral $\int_C \frac{2z+1}{z^2+z}$ dz = of the form $\frac{f'(z)}{f(z)}$.

Put $z^2 + z = 0$

∴ z =0, -1

Res [$\frac{2z+1}{z^2+z}$, 0] = 1 and Res [$\frac{2z+1}{z^2+z}$, 0] = 1

Hence $\frac{1}{2\pi i}\int_C \frac{2z+1}{z^2+z}$ dz = 2πi(1+1)= 4πi

Table 1: Comparison of different methods of complex integrals

Test	Conditions	Outcome
Direct evaluation of line integrals	Given function to be integrated is evaluated along a line or a curve	Result will be found from directly calculating the integral or changing the curve in to parametric form.
Darboux Property	Given function f(z) (i) is continuous on a contour L of length l. (ii) |f(z)| ≤ M	$\|\int_L f(z)dz\| \leq$ Ml
Cauchy's Goursat Theorem	Given function f(z) (i) is analytic (ii) The contour C is closed and continuous	$\int_C f(z)dz = 0$
Cauchy's Integral Formula	Given function is (i) of the type $\frac{f(z)}{(z-z_0)(z-z_1).....(z-z_n)}$ (ii) f(z) is analytic	$\int_C \frac{f(z)}{z-z_0}$ dz = $2\pi i$ [f(z_0)+ f(z_1)+---+ f(z_n)]
Derivative of analytic function	Given function to be integrated is (i) of the form $\frac{f(z)}{(z-z_0)^{n+1}}$. (ii) f(z) should be analytic.	$\int_C \frac{f(z)}{(z-z_0)^{n+1}}$ dz = $\frac{2\pi i}{n!} f^{(n)}(z_0)$
The Argument principle	Given function to be integrated is (i) of the form $\frac{f'(z)}{f(z)}$	$\int_C \frac{f'(z)}{f(z)}$ dz = $\frac{1}{2\pi i}$ [N-P]
Cauchy's Residue Theorem	Given function to be integrated (i) is regular except at finite number of poles	$\int_C f(z)dz$ = 2πi $\sum R$ where $\sum R$ = sum of residues of f(z) at its poles with in C

3. Conclusion
Although there is no universal criterion for evaluating complex integrals as several methods are applicable to solve one problem. But with the correct selection of properties or theorems for evaluating complex integrals, one can get the answers very quickly and easily and in a shortcut way.

4. References

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